On the general Gauss-Bonnet theorem

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Abstract

The Gauss-Bonnet theorem for an even dimensional, compact, oriented, smooth manifold M says that the integral over M of a representative of the Euler class of the tangent bundle equals the Euler characteristic of M. In this paper, we develop a proof of this theorem, while also giving a brief overview of the concepts that are involved. The main steps of our proof are given in [1] , as a series of exercises.

1 Introduction

The Gauss-Bonnet is one of the most celebrated theorems in differential geometry; there is a whole wealth of formulations and proofs in the literature. A survey paper on this theorem is [2]. Besides its intrinsic beauty, it motivates incursions into other interesting areas of geometry/topology, such as index theory or characteristic classes. When examined in the light of the latter, Gauss-Bonnet may be turned into a statement that belongs to algebraic topology: if one uses singular (co)homology to define the Euler class, Gauss-Bonnet says that the Euler class of the tangent bundle of a compact, connected $Z-$oriented manifold M, capped with the orientation class $\omega \in H_{dimM}(M)$, gives the Euler characteristic of M. In this paper we will be using smooth forms and deRham cohomology instead of singular and hence the Euler class will be defined in the spirit of the so-called Chern-Weil theory, which directly produces classes in the cohomology of the manifold from a metric connection on the tangent bundle.

Our approach is to first prove the theorem for the case of spheres by direct computation; second, an explicit orientation class will be constructed for the associated sphere bundle of the tangent bundle τ_M of M, and further we will relate this class to the Euler class of τ_M.

The paper is divided into three main sections. In section 2, we review some of the concepts we will be dealing with, following [1]; in section 3, we prove the theorem for spheres and in the last section we go on to proving the general case.

Alternative modern proofs of Gauss-Bonnet include a heat-equation approach, and the superconnections approach of Mathai-Quillen.

2 Some general concepts

2.1 Connections and curvature on vector bundles

We first introduce the notion of bundle valued differential forms.

Definition Given a vector bundle $\xi \to M^n$, a ξ-valued p-form on M is a section of the bundle $\Omega^p(\tau_M, \xi)$. The fiber of the latter at $x \in M$ consists of the skew-symmetric p-linear maps $(\tau_M)_x \times \ldots \times (\tau_M)_x \longrightarrow \xi_x$.

We denote by $\Omega^p(M, \xi)$ the $\Omega^0(M)$-module of ξ-valued p-forms. Note that $\Omega^0(M, \xi)$ is just the module of smooth sections. Letting $\Omega(M) = \sum \Omega^p(M)$ and $\Omega(M, \xi) = \sum \Omega^p(M, \xi)$, $\Omega(M, \xi)$ is a graded left module over the graded algebra $\Omega(M)$, the module multiplication being given by

$$(\Phi \wedge \Omega)_x(h_1 \ldots h_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \epsilon_\sigma \Phi_x(h_{\sigma(1)} \ldots h_{\sigma(p)}) \Omega_x(h_{\sigma(p+1)} \ldots h_{\sigma(p+q)})$$

where $\Phi \in \Omega^p(M)$, $\Omega \in \Omega^q(M, \xi)$, $x \in M$, $h_i \in (\tau_M)_x$.

The $\Omega^0(M)$-bilinear map $(\Phi, s) \longrightarrow \Phi \wedge s$, $\Phi \in \Omega(M)$, $s \in Sec(\xi)$ induces an isomorphism of graded $\Omega(M)$ modules: $\Omega(M) \otimes_{\Omega^0(M)} Sec(\xi) \xrightarrow{\simeq} \Omega(M, \xi)$.

Definition A connection on the real vector bundle $\xi \longrightarrow M$ is an $\mathbf{R}$-linear map $\bigtriangledown : \Omega^0(M, \xi) \longrightarrow \Omega^1(M, \xi)$ satisfying the Leibniz rule: $\bigtriangledown(fs) = df \wedge s + f \bigtriangledown s$, $f \in \Omega^0(M)$, $s \in \Omega^0(M, \xi)$.

A general fact is that any vector bundle admits a connection. Locally, a connection is specified by a matrix of 1-forms $A_{ij} \in \Omega^1(U)$, $U \subset M$ being an open set over which the bundle trivializes. This implies that there are sections $e_1 \ldots e_k$ such that $e_1(p) \ldots e_k(p)$ is a basis of ξ_p; write $\bigtriangledown(e_i) = \sum A_{ij} \otimes e_j$.

Alternatively, we can think of the connection locally as an element of $\Omega^1(Hom(\xi_{/U}, \xi_{/U}))$ written as $\bigtriangledown = A_{ij} \otimes e_j \otimes e_i^*$, where e_i^* are the dual sections to e_i. If the e_i's form an orthonormal frame at each point, then $\bigtriangledown e_i = A_{ij} \otimes e_j$; in general $\bigtriangledown s = A_{ij} \otimes e_i(s) e_j$.

A connection $\bigtriangledown$ extends to a map $\bigtriangledown : \Omega^p(M, \xi) \longrightarrow \Omega^{p+1}(M, \xi)$, by

$$\bigtriangledown \Psi(X_0 \ldots X_p) = \sum_{j=0}^{p} (-1)^j \bigtriangledown_{X_j} (\Psi(X_0 \ldots \hat{X}_j \ldots X_p)) +$$

$$\sum_{i<j} (-1)^{i+j} \Psi([X_i, X_j] X_0 \ldots \hat{X}_i \ldots \hat{X}_j \ldots X_p)$$

, $\Psi \in \Omega^p(M, \xi)$, X_i vector fields on M. Here $\bigtriangledown_X = i(X) \circ \bigtriangledown$ and $i(X)$ denotes the "interior product" operator, $i(X) : \Omega^p(M, \xi) \longrightarrow \Omega^{p-1}(M, \xi)$, $i(X)s = 0$ by definition, $(i(X)\Omega)_x(h_1 \ldots h_{p-1}) = \Omega_x(X(x), h_1 \ldots h_{p-1})$, $x \in M$, $h_i \in (\tau_M)_x$,

$\Omega \in \Omega^p(M,\xi)$. The extended connection satisfies the relation $\nabla(\Phi \wedge \Psi) = d\Phi \wedge \Psi + (-1)^{deg\Phi}\Phi \wedge \nabla\Psi$, $\Phi \in \Omega^p(M)$, $\Psi \in \Omega^p(M,\xi)$.

The composition $F = \nabla \circ \nabla : \Omega^0(M,\xi) \longrightarrow \Omega^2(M,\xi)$ turns out to be $\Omega^0(M)-$linear. Also $F^\nabla \in Hom_{\Omega^0(M)}(\Omega^0(M,\xi), \Omega^2(M,\xi)) \simeq$ $Hom_{\Omega^0(M)}(\Omega^0(M,\xi), \Omega^0(M,\xi)) \otimes_{\Omega^0(M)} \Omega^2(M) \simeq \Omega^0(M, Hom(\xi,\xi)) \otimes_{\Omega^0(M)} \Omega^2(M)$ $\simeq \Omega^2(M, Hom(\xi,\xi))$. The first isomorphism expresses the adjointness of Hom and $\otimes$, the proof of the second, given in [1] is more involved and would lead us too far astray from the purpose of this paper. The 2-form F^∇ is called the curvature of (ξ,∇).

We write F^∇ locally:

$$\nabla \circ \nabla(e_i) = \sum dA_{ij} \otimes e_j - \sum A_{ij} \wedge \nabla(e_j) = \sum dA_{ij} \otimes e_j - \sum A_{ij} \wedge \sum A_{jv} \otimes e_v =$$

$$= \sum_v (dA_{iv} \otimes e_v - (\sum_j A_{ij} \wedge A_{jv}) \otimes e_v),$$

so $F^\nabla(e_i) = \sum_v (dA - A \wedge A)_{iv} \otimes e_v$. The matrix $dA - A \wedge A$ is called the curvature matrix of the given connection.

In what follows, we explain how the connection ∇ on ξ induces a connection on ξ^* and $\xi \otimes \xi^* \simeq Hom(\xi,\xi)$.

We claim that there is a unique connection ∇^* on ξ^* such that $< \nabla^* s^*, s >$ $+ < s^*, \nabla s > = d < s, s^* >$, $s^* \in Sec(\xi^*)$, $s \in Sec(\xi)$. Indeed, fixing a $s^* \in Sec(\xi^*)$, a simple computation shows that the map $Sec(\xi) \longrightarrow \Omega^1(M)$ given by $s \longrightarrow d < s^*, s > - < s^*, \nabla s >$ is $\Omega^0(M)$ linear. Hence there is a unique element $\nabla^* s^* \in \Omega^1(M,\xi^*)$, such that $< \nabla^* s^*, s > = d < s^*, s > - < s^*, \nabla s >$. The assignment $s^* \longrightarrow \nabla^* s^*$ defines a connection on ξ^*, which is the only one satisfying the desired relation. A similar approach shows that given connections ∇_1, ∇_2 on ξ_1, ξ_2, respectively, there is a unique connection $\tilde{\nabla}$ on $\xi_1 \otimes \xi_2$ such that $\tilde{\nabla}(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$, $s_i \in Sec(\xi_i)$.

A computation that can be easily carried out by working over local orthonormal frames shows that $\tilde{\nabla} F^\nabla = 0$, where $\tilde{\nabla}$ is the induced connection on $\Omega^*(M, Hom(\xi,\xi))$. This equality is known as the second Bianchi identity.

Since the difference of two connections on ξ is an element of $\Omega^1(M, Hom(\xi,\xi))$ and locally exterior differentiation is a legitimate connection, we can write $\nabla = d + \Omega$, $\Omega \in \Omega^1(U, Hom(\xi_{/U}, \xi_{/U}))$, whre U is an open set is an open set over which the bundle trivializes. A set of computations that we will not carry out shows that $\tilde{\nabla} = d + L_\Omega$ locally, where $\tilde{\nabla}$ is the induced connection on $\Omega^*(M, Hom(\xi,\xi))$ and L_Ω represents the Lie bracket with respect to Ω.

The above considerations show that the second Bianchi identity can be written locally as $dF^\nabla = F^\nabla \wedge \Omega - \Omega \wedge F^\nabla$. This equality will be used in section **2.3**.

2.2 The Pfaffian

The Pfaffian is a homogeneous polynomial of degree n in $n(2n-1)$ real variables, or alternatively a polynomial in the skew-symmetric $2n \times 2n$ matrices A_{ij}. For such a matrix A_{ij}, we let $\omega(A) = \sum_{i<j} A_{ij} e_i \wedge e_j \in \wedge^2(\mathbf{R}^n)$ and define $Pf(A)$ from the equality $\omega(A) \wedge \ldots \wedge \omega(A) = n! Pf(A) e_1 \wedge e_2 \ldots \wedge e_{2n}$.

An important relation that the Pfaffian satisfies, proved in [1] and [3] is $Pf(BAB^t) = Pf(A) detB$ (2), where A is a $2n \times 2n$ skew-symmetric matrix and B is arbitrary. From this, we conclude that the Pfaffian is invariant under the group SO_{2n}: $Pf(B^{-1}AB) = Pf(A)$, $(\forall) B \in SO_{2n}$.

Another important observation is that $Pf(A)$ can be defined as long as the entries of A are in some commutative algebra $\mathcal{A}$ over $\mathbf{R}$. Then (2) still holds. Indeed, consider the ring $\mathbf{R}[A_{ij}, B_{ij}]$ obtained by adjoining commuting indeterminates A_{ij} and B_{ij} to $\mathbf{R}$. Then $Pf(BAB^t)$ and $detBPf(A)$ are still in $\mathbf{R}[A_{ij}, B_{ij}]$. (2) tells us that these polynomials have the same values on all $a_{ij}, b_{ij} \in \mathbf{R}$. Therefore they are equal as polynomials in the indeterminates A_{ij}, B_{ij}.

It is straightforward to see, from the definition and the remarks in the preceding paragraph that the Pfaffian of commuting variables $X_{ij}, 1 \leq i < j \leq 2n$ can also be written:

$$\frac{1}{n!} \sum_{\sigma \in S_{2n}, \sigma(2v-1) < \sigma(2v)} \epsilon(\sigma) \prod_{v=1}^{n} X_{\sigma(2v-1)\sigma(2v)}.$$

Here $\epsilon(\sigma)$ is the signature of σ.

Furthermore, by letting $X_{ij} = \epsilon_i \wedge \epsilon_j, i < j$, where $\epsilon_1 \ldots \epsilon_{2n}$ is the standard basis of $Alt^1(\mathbf{R}^{2n})$, we get that

$$Pf(\epsilon_i \wedge \epsilon_j) = 1 * 3 * 5 \ldots (2n-1) \epsilon_1 \wedge \ldots \wedge \epsilon_{2n}.$$

(There are $\binom{2n}{2}\binom{2n-2}{2} \ldots \binom{2}{2} = \frac{(2n)!}{2^n}$ permutations as above and we are dividing by $n!$; $\epsilon(\sigma)$ gets multiplied by itself, so it cancels.) We will use this formula in our proof of the theorem for even-spheres.

2.3 The Euler Class

The notion of inner product (metric) on an arbitrary vector bundle is a natural generalization of the Riemannian metric.

Definition For a bundle $\xi \to M$, a connection $\bigtriangledown$ on $(\xi, <,>)$ is said to be metric if $d < s_1, s_2 > = < \bigtriangledown s_1, s_2 > + < s_1, \bigtriangledown s_2 >$ for any two sections.

In what follows, we will be considering a vector bundle $\xi \to M$ of fibre dimension $2k$. Let $e_1 \ldots e_{2k} \in \Omega^0(\xi/U)$, for some open set U, such that $e_1(p) \ldots e_{2k}(p)$ forms an orthonormal basis of ξ_p, for all p. Let A_{ij} be the connection matrix of 1-forms, for some metric connection on ξ. Since $< e_i, e_k >$ is constant, we get:

$$0 = < \sum A_{ij} \otimes e_j, e_k > + < e_i, \sum A_{kj} \otimes e_j > =$$

$$= \sum_j A_{ij} < e_j, e_k > + \sum_j A_{kj} < e_i, e_j > = A_{ik} + A_{ki}$$

so A_{ij} is skew-symmetric.

The curvature matrix of 2-forms with respect to the orthonormal frame $\mathbf{e}$ will be given locally by a $2k \times 2k$ matrix with entries in $\Omega^2(U)$: $F^{\bigtriangledown}(\mathbf{e}) = dA - A \wedge A$. Since A is skew-symmetric, we see that the same happens for $F^{\bigtriangledown}(\mathbf{e})$, hence by the discussion in 2.2, we can take $Pf(F^{\bigtriangledown}(\mathbf{e})) \in \Omega^{2k}(U)$.

In another orthonormal frame $\mathbf{e}'$ over U, $F^{\bigtriangledown}(\mathbf{e}')_p = B_p F^{\bigtriangledown}(\mathbf{e}) B_p^{-1}$, where B_p is the orthogonal transition matrix between $\mathbf{e}(p)$ and $\mathbf{e}'(p)$.

If we assume further that the vector bundle ξ is oriented, one of the various (equivalent) notions of orientation says that the structure group can be taken to be SO_{2k}. If $\mathbf{e}(p)$ and $\mathbf{e}'(p)$ are oriented orthonormal bases for $\xi_p, p \in U, B_p \in SO_{2k}$, from section 2.2 we get $Pf(F^{\bigtriangledown}(\mathbf{e})) = Pf(F^{\bigtriangledown}(\mathbf{e}'))$, so $Pf(F^{\bigtriangledown})$ becomes a well-defined $2k-$form on M.

A nontrivial fact about this form is that it is closed. To prove closedness, we recall the local form of Bianchi's second identity: $dF^{\bigtriangledown} = F^{\bigtriangledown} \wedge \Omega - \Omega \wedge F^{\bigtriangledown}$. We denote by Pf' the (antisymmetric) matrix of partial derivatives of the Pfaffian. Then $dPf(F^{\bigtriangledown}) = \text{Trace}(Pf'(F^{\bigtriangledown})^t \wedge dF^{\bigtriangledown})$.

For any antisymmetric matrix B, $Pf'(B)^t \cdot B = B \cdot Pf'(B)^t$. Indeed, by differentiating the identity $(PfB)^2 = \det B$, we get $2Pf(B) \cdot Pf'B = \det' B$. Transposing, we get $2Pf(B) \cdot Pf'(B)^t = \det' B^t = B^*$, the adjoint matrix of B. Multiplying by B, we get $PfB \cdot Pf'(B)^t \cdot B = PfB \cdot B \cdot Pf'(B)^t$. It follows that $Pf'(B)^t \cdot B = B \cdot Pf'(B)^t$, when B is nonsingular; a continuity argument shows that the above identity always holds.

In particular $Pf'(F^{\bigtriangledown})^t \wedge F^{\bigtriangledown} = F^{\bigtriangledown} \wedge Pf'(F^{\bigtriangledown})^t$.

Hence $dPf(F^{\bigtriangledown}) = \text{Trace}(Pf'(F^{\bigtriangledown})^t \wedge F^{\bigtriangledown} \wedge \Omega - Pf'(F^{\bigtriangledown})^t \wedge \Omega \wedge F^{\bigtriangledown})$

$$\text{Trace}(F^{\bigtriangledown} \wedge [Pf'(F^{\bigtriangledown})^t \wedge \Omega] - [Pf'(F^{\bigtriangledown})^t \wedge \Omega] \wedge F^{\bigtriangledown}) = 0.$$

Another crucial property of $Pf(F^{\bigtriangledown})$ is summarized in the following theorem, proved in [1]:

Theorem 1 The cohomology class determined by $Pf(F^{\bigtriangledown})$ is independent of the choice of metric on ξ and of the compatible metric connection.

Finally, we define the Euler class:

Definition The Euler class of the oriented, real, $2k-$dimensional vector bundle ξ is the cohomology class $e(\xi) = \left[Pf\left(\frac{-F^{\bigtriangledown}}{2\pi}\right)\right] \in H^{2k}(M)$.

3 Gauss-Bonnet for spheres

In this section we are going to prove the Gauss-Bonnet theorem, $\int_M Pf\left(\frac{-F^{\bigtriangledown}}{2\pi}\right) = \chi(M^{2n})$, M even-dimensional, compact, oriented, smooth manifold, for the case when M is a sphere S^{2n}, endowed with the standard euclidian metric and the Levi-Civita connection. In this case $\chi(S^{2n}) = 2$. The result will follow from the following 4 Lemmas:

Lemma 1 Any sphere $S^n \subset \mathbf{R}^{n+1}$, $n \geq 2$, with the induced euclidian metric, has constant sectional curvature equal to 1

Proof I'll add this later, hopefully smth that is not computations with Christoffel symbols in the stereographic charts.

Lemma 2 If $sec(\pi) = k$ for all 2-planes in T_pM, then $R(v_1, v_2)v_3 = k(< v_2, v_3 > v_1 - < v_1, v_3 > v_2), \forall v_1, v_2, v_3 \in T_pM$.

Proof Introduce the multilinear maps:

$$R_k(v_1, v_2)v_3 = k(< v_2, v_3 > v_1 - < v_1, v_3 > v_2)$$

$$R_k(v_1, v_2, v_3, v_4) = k(< v_2, v_3 >< v_1, v_4 > - < v_1, v_3 >< v_2, v_4 >).$$

Then one can see that $R_k(v_1, v_2, v_3, v_4)$ satisfies the same symmetry relations as $R(X, Y, Z, W) =< R(X, Y)Z, W >$, namely:

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z) \qquad (1)$$

$$R(X, Y, Z, W) = R(Z, W, X, Y) \qquad (2)$$

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0 \qquad (3)$$

Then $T(v_1, v_2, v_3, v_4) = R(v_1, v_2, v_3, v_4) - R_k(v_1, v_2, v_3, v_4)$ is a tensor satisfying the same symmetries and $sec(\pi) = k, \forall \pi \in T_pM$ implies $T(v, w, w, v) = 0, \forall v, w \in T_pM$.

For $w = w_1 + w_2$, we get:

$$0 = T(v, w_1 + w_2, w_1 + w_2, v) = T(v, w_1, w_2, v) + T(v, w_2, w_1, v) =$$

$$= 2T(v, w_1, w_2, v) = -2T(v, w_1, v, w_2).$$

Properties (1) and (2) will then imply that T is alternating in all 4 variables, and then by (3), $T \equiv 0$, q.e.d.

One can also obtain the desired equality by using the identity that shows how the sectional curvatures determine the whole curvature tensor. This approach is used in [7].

Lemma 3 Given a smooth manifold M^n, let $e_1 \dots e_n$ be an orthonormal frame on some open U. Let θ^i be the dual coframe of 1-forms. Then the curvature matrix of 2-forms is given locally by $\Omega_{ij} = -1/2 \sum_{k,l} R^i_{jkl} \theta^k \wedge \theta^l$, where $R^i_{jkl} =< R(e_k, e_l)e_j, e_i >$.

Proof We will be using a standard summation convention, namely when an index appears on one side of an expression and does not appear on the other, we are summing after it. Let $\bigtriangledown e_i = \omega_{ij} \otimes e_j$, $[e_i, e_j] = c_{ij}^p e_p$. Then

$$\bigtriangledown_{e_k} e_i = \omega_{ij}(e_k) e_j = < \omega_{ij}, \theta^k > e_j = \Gamma_{ik}^j e_j.$$

Also $\Gamma_{ik}^j = -\Gamma_{jk}^i$, since ω_{ij} is skew-symmetric. To prove the claimed equality, it suffices to check that both sides, when evaluated on pairs of vector fields (e_s, e_t), give the same result.

$$R_{jkl}^i e_i = R(e_k, e_l) e_j = \bigtriangledown_{e_k} \bigtriangledown_{e_l} e_j - \bigtriangledown_{e_l} \bigtriangledown_{e_k} e_j - \bigtriangledown_{[e_k, e_l]} e_j =$$

$$\bigtriangledown_{e_k}(\Gamma_{jl}^s e_s) - \bigtriangledown_{e_l}(\Gamma_{jk}^s e_s) - \bigtriangledown_{c_{kl}^s e_s} e_j =$$

$$\Gamma_{jl}^s \bigtriangledown_{e_k} e_s + e_k(\Gamma_{jl}^s) e_s - \Gamma_{jk}^s \bigtriangledown_{e_l} e_s - e_l(\Gamma_{jk}^s) e_s - c_{kl}^s \bigtriangledown_{e_s} e_j =$$

$$\Gamma_{jl}^s \Gamma_{sk}^t e_t - \Gamma_{jk}^s \Gamma_{sl}^t e_t + e_k(\Gamma_{jl}^s) e_s - e_l(\Gamma_{jk}^s) e_s - c_{kl}^s \Gamma_{js}^t e_t.$$

Evaluating the right side on vector fields, we have $-\frac{1}{2} \sum_{k,l} R_{jkl}^i \theta^k \wedge \theta^l (e_s, e_t) = -\frac{1}{4}(R_{jst}^i - R_{jts}^i) = -\frac{1}{2} R_{jst}^i = -\frac{1}{2}(\Gamma_{jt}^v \Gamma_{vs}^i - \Gamma_{js}^v \Gamma_{vt}^i + e_s(\Gamma_{jt}^i) - e_t(\Gamma_{js}^i) - c_{st}^v \Gamma_{jv}^i)$ $(\star)$

The next step is to compute $\Omega_{ij}(e_s, e_t)$. We have:

$$d\omega_{ij} = d\Gamma_{ik}^j \wedge \theta^k + \Gamma_{ik}^j d\theta^k = e_l(\Gamma_{ik}^j) \theta^l \wedge \theta^k + \Gamma_{ik}^j d\theta^k \Longrightarrow$$

$$d\omega_{ij}(e_s, e_t) = \frac{1}{2} e_s(\Gamma_{it}^j) - \frac{1}{2} e_t(\Gamma_{is}^j) - \frac{1}{2} \Gamma_{ik}^j \theta^k([e_s, e_t])$$

$$d\omega_{ij}(e_s, e_t) = \frac{1}{2} e_s(\Gamma_{it}^j) - \frac{1}{2} e_t(\Gamma_{is}^j) - \frac{1}{2} \Gamma_{ik}^j c_{st}^k \quad (\star\star)$$

The last term in the last but one expression was obtained from the formula $2(d\omega)(X_1, X_2) = X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])$.

Finally, $-\omega_{iv} \wedge \omega_{vj}(e_s, e_t) = \frac{1}{2}(\Gamma_{vs}^j \Gamma_{it}^v - \Gamma_{is}^v \Gamma_{vt}^j)$. From this, and the relations $(\star)$ and $(\star\star)$, the conclusion follows, keeping in mind that $\Gamma_{ik}^j = -\Gamma_{jk}^i$.

Lemma 4 The volume of S^{2n} is given by $Vol(S^{2n}) = \frac{2^{2n+1} n! \pi^n}{(2n)!}$.

Proof Define an $(n-1)$-form $\omega_0 \in \Omega^{n-1}(\mathbf{R}^n)$ by

$$\omega_{0x}(w_1 \ldots w_{n-1}) = det(x, w_1 \ldots w_{n-1}) \in Alt^{n-1}(\mathbf{R}^n),$$

for $x \in \mathbf{R}^n$. Since $\omega_{0x}(e_1 \ldots \hat{e_i} \ldots e_n) = (-1)^{i-1} x_i$, we have

$$\omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n.$$

If $x \in S^{n-1}$ and $w_1 \ldots w_{n-1}$ is a basis of $T_x S^{n-1}$, then $x, w_1 \ldots w_{n-1}$ becomes a basis for $\mathbf{R}^n$ and the first formula in this proof shows that $\omega_{0x} \neq 0$. Hence

$\omega_{0/S^{n-1}}$ is an orientation form on S^{n-1}. For the orientation of S^{n-1} given by ω_0, the basis $w_1 \dots w_{n-1}$ of T_xS^{n-1} is positively oriented if and only if the basis $x, w_1 \dots w_{n-1}$ for $\mathbf{R}^n$ is positively oriented. Hence S^{n-1} with the induced metric has volume form $\omega_{0/S^{n-1}}$.

Now the volume of S^{n-1} can be calculated by applying Stokes' theorem to D^n with the standard orientation of $\mathbf{R}^n$ and the $(n-1)$-form ω_0 on $\mathbf{R}^n$. Since $d\omega_0 = n dx_1 \wedge \dots \wedge dx_n$, we have that

$$Vol(S^{n-1}) = \int_{S^{n-1}} \omega_0 = \int_{D^n} d\omega_0 = nVol(D^n).$$

The latter is just computing an iterated integral, and turns out to be $Vol(D^{2n+1}) = \frac{2^{2n+1} n! \pi^n}{(2n+1)!}$. This yields the desired expression for the volume of the sphere.

From Lemma 1 and Lemma 2, we deduce $R^i_{jkl} = \delta_{lj}\delta_{ki} - \delta_{kj}\delta_{li}$. The only cases when R^i_{jkl} is not zero are $l = j, k = i, i \neq j$ and $k = j, l = i, i \neq j$. It follows from Lemma 3 that the curvature matrix is locally $\Omega_{ij} = -\theta^i \wedge \theta^j$. The last formula from section 2.2, combined with Lemma 4 and the observation that $\chi(S^{2n}) = 2$, show the validity of Gauss-Bonnet for even spheres.

4 The General Case

4.1 Various Notions of Orientation

The sphere bundle of a smooth manifold (M^n, g) is defined as follows: take the direct sum between τ_M and the trivial line bundle over M; this new vector bundle has a metric induced by g; the unit spheres in its fibers will be the fibers of a S^n-bundle over M, the associated sphere bundle, which we denote by $S(\tau_M \oplus 1)$. We make the convention that the zero section in this bundle is $s_0(p) = (0, -1)$ and the section at infinity, $s_\infty(p) = (0, 1)$.

Our aim in what follows is to establish the following implication:

$$\mathbf{M\ oriented} \Longrightarrow \tau_{\mathbf{M}}\ \mathbf{oriented} \Longrightarrow \mathbf{S}(\tau_{\mathbf{M}} \oplus \mathbf{1})\ \mathbf{oriented}.$$

By τ_M being oriented we mean that all transition maps have positive determinant. This is the same as requiring τ_M to have structure group SO_n, keeping in mind that the structure group of any real vector bundle can be reduced to O_n. Furthermore, a sphere bundle $S^n \to E \to M$ is defined to be orientable if there is a class $[\omega] \in H^n(E)$, which generates the n-th cohomology of the fiber, when restricted to it.

The first implication is just a matter of checking definitions. Choose a trivializing atlas (U_α, ϕ_α) on M such that the determinants of the jacobians of the transition maps are positive. A diffeomorphism between $\pi^{-1}(U_\alpha)$ and

$U_\alpha \times \mathbf{R^n}$ is given by $(p, v_p) \xrightarrow{\psi_\alpha} (\phi_\alpha(p), f_1, \dots, f_n)$, where f_i are the components of v_p with respect to the standard coordinate vector fields. Then $(U_\alpha \cap U_\beta) \times \mathbf{R^n} \overset{\psi_\alpha \psi_\beta^{-1}}{\longrightarrow} (\mathbf{U}_\alpha \cap \mathbf{U}_\beta) \times \mathbf{R^n}$ will be given fiberwise by the jacobian of $\phi_\alpha \phi_\beta^{-1}$: $\psi_\alpha \psi_\beta^{-1}(p, v) = (p, J(\phi_\alpha \phi_\beta^{-1})|_{\phi_\beta(p)})$, which is positive by the assumption that M is oriented.

The proof of the second implication is more involved . We will follow ideas from [4], rather then the "induction on open sets" argument given in [1].

Since the transition maps of τ_M assume values in SO_n, those of $\tau_M \oplus 1$ are in SO_{n+1}. We first prove that it is possible to find a cover of the total space E of the associated sphere bundle and classes $[\sigma_U] \in H^n(E_{/U})$ that restrict to generators of $H^n(F_x)$ for $x \in U$ and for all U in the cover.

Fix a generator σ of $H^n(S^n)$ and a trivialization of E so that the transition maps $g_{\alpha\beta}$ take values in SO_{n+1}. Let $\rho_\alpha : U_\alpha \times S^n \to S^n$ be the projection and let $\pi^{-1}(x)$ be a fiber of $S(\tau_M \oplus 1)$. Define $[\sigma_\alpha] \in H^n(E_{/U})$ by $[\sigma_\alpha] = {\Phi_\alpha}^* {\rho_\alpha}^* [\sigma]$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \overset{\simeq}{\to} U_\alpha \times S^n$. Write $[\sigma_\alpha]_{/X}$ and $\Phi_{\alpha/X}$ for the restrictions $[\sigma_\alpha]_{/\pi^{-1}(x)}$ and $\Phi_{\alpha/\pi^{-1}(x)}$, respectively.

Then for $x \in U_\alpha$, $[\sigma_\alpha]_{/X} = (\Phi_{\alpha/X})^*[\sigma]$. The fact that the above defined classes agree on intersections is equivalent to saying:

$$[\sigma] = ((\Phi_{\beta/X})^*)^{-1} (\Phi_{\alpha/X})^* [\sigma] = (\Phi_{\alpha/X} \circ \Phi_{\beta/X}^{-1})^* [\sigma] = g_{\alpha\beta}(x)^* [\sigma], x \in U_\alpha \cap U_\beta.$$

The latter is true since $g := g_{\alpha\beta}(x)$ is an orientation preserving diffeomorphism of S^n, so $\int_{S^n} g^*[\sigma] = \int_{g(S^n)} [\sigma] = \int_{S^n} [\sigma] = 1$.

To show that the $[\sigma_\alpha]'s$ patch to define a global class, we have to introduce more machinery. The proofs of the results we are going to state and use can be found in [4].

Given a countable open cover (U_α) of a manifold M, the chain with coefficients in the sheaf of smooth p-forms is the following chain:

$$0 \to \Omega^p(M) \xrightarrow{r} \prod \Omega^p(U_0) \xrightarrow{\delta} \prod \Omega^p(U_0 \cap U_1) \xrightarrow{\delta} \prod \Omega^p(U_0 \cap U_1 \cap U_2) \to \dots.$$

An element of $\prod \Omega^p(U_1 \cap \dots \cap U_k)$ is specified by a p-form on each k-intersection. (On intersections with a repeated index, we consider the 0-form; if the index sets differ by a permutation, the forms assigned to them differ by the sign of that permutation.) The first differential, r, is just restriction. The differential of an element $\omega \in \prod \Omega^p(U_{i_1} \cap \dots \cap U_{i_q})$ with "components" $\omega_{i_1 \dots i_q}$ is given by $(\delta\omega)_{j_1 \dots j_{q+1}} = \sum_{i=0}^{q+1} (-1)^i \omega_{j_1 \dots \hat{j_i} \dots j_{q+1}}$.

It is immediate to check $\delta^2 = 0$, so the δ-cohomology of the above complex is defined. A first nontrivial result is that the cohomology groups vanish in all dimensions.

We wish to relate the deRham groups of M to the cohomology groups of a double complex which we define in what follows. Let $K^{p,q} = \prod \Omega^q(U_0 \cap \dots \cap U_p)$. Keeping the first (second) index fixed, we get a chain with differential

$d(\delta)$. Denoting $K^n = \sum_{p+q=n} K^{p,q}, K^n$ is a chain complex (the "Cech-deRham complex") with differential $D = \delta + (-1)^p d$. Note that $K^{*,q}$ is an augmented complex (by the group $\Omega^q(M)$). The main result we will use to prove the existence of a global orientation class is the following theorem:

Theorem 2 The double complex K computes the deRham cohomology of M, more precisely the restriction map $r : \Omega^* \to K^*$ is a chain map inducing an isomorphism in cohomology. More generally, if all the rows (columns) of an augmented complex are exact, then its cohomology is isomorphic to the cohomology of the initial column (row).

We finally proceed to prove the existence of a global orientation class on E, the total space of $S(\tau_M \oplus 1)$. We will use the fact that any (compact) manifold has a (finite) good cover, that is, a cover such that all nonempty intersections are contractible. In fact, it is true that any open cover has a refinement which is a good cover. This, together with the observation that (U_α) good cover of M implies $\pi^{-1}(U_\alpha)$ good cover of E, implies that we can choose local orientation classes $[\sigma_\alpha] \in H^n(E_{/U_\alpha})$. By abuse of notation, we will write $\Omega^n(U_\alpha \cap U_\beta)$ instead of $\Omega^n(E_{/(U_\alpha \cap U_\beta)}$.

The σ_α's define an element $\sigma^{0,n}$ in $\prod \Omega^n(U_\alpha)$. By the theorem, to prove that $[\sigma_\alpha]$ patch to define a global class, it suffices to show that $\sigma^{0,n}$ extends to a $D-$cocycle in K^n.

At this step, it is useful to have in mind the following picture:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n(M) \arrow[r, "r"] & \prod \Omega^n(U_\alpha) \ni \sigma^{0,n} \arrow[r, "\delta"] & \cdots & & \cdots\cdots & \\
& & \vdots \arrow[u, "d"] & \cdots\ \prod \Omega^{n-1}(U_\alpha \cap U_\beta) \ni \sigma^{1,n-1} & & \cdots\cdots & \\
& & \vdots & \vdots \arrow[u, "-d"] & & & \\
0 \arrow[r] & \Omega^1(M) \arrow[r, "r"] & \prod \Omega^1(U_\alpha) \arrow[u, "d"] \arrow[r, "\delta"] & \vdots & & \cdots\cdots & \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "r"] & \prod \Omega^0(U_\alpha) \arrow[u, "d"] \arrow[r, "\delta"] & \prod \Omega^0(U_\alpha \cap U_\beta) \arrow[u, "-d"] & \cdots \arrow[r, "\delta"] & \prod \Omega^0(U_{\alpha_1} \cap \ldots \cap U_{\alpha_{n+1}}) \ni \sigma^{n,0} \arrow[u, "(-1)^n d"] \arrow[r, "\delta"] & {}
\end{tikzcd}$$

To extend $\sigma^{0,n}$ to a $D-$cocycle is to find elements $\sigma^{1,n-1}, \ldots, \sigma^{n,0}$, belonging to the groups specified in the picture, such that:

$$\delta\sigma^{0,n} = d\sigma^{1,n-1}, \delta\sigma^{1,n-1} = -d\sigma^{2,n-2} \ldots \delta\sigma^{n-1,1} = (-1)^{n-1} d\sigma^{n,0}, \delta\sigma^{n,0} = 0.$$

All but the last of the above equalities are trivially satisfied, since the columns in the picture are exact, by our choice of good covers. Thus $D(\sigma^{0,n} + \sigma^{1,n-1} + \ldots + \sigma^{n,0}) = \delta\sigma^{n,0}$. Also $d(\delta\sigma^{n,0}) = \delta(d\sigma^{n,0}) = \pm\delta(\delta\sigma^{n-1,1}) = 0$ and this implies $\delta\sigma^{n,0} = 0$ by injectivity of the first vertical differential.

4.2 The vertical bundle and another correspondence between orientations

Lemma 5 Let $\xi = (E, \pi, B, F)$ be a vector bundle. If there is a nowhere vanishing section $\Delta \in$Sec$(\wedge^r \xi^*)$, (ξ^* is the dual bundle) then the vector bundle is orientable.

Proof If such a section exists, then $\Delta(x)$ orients F_x in the sense of linear algebra, i.e. specifies whether a basis is positive or negative. Let (U_α, ψ_α) be a trivialization, $\psi_\alpha : U_\alpha \times F \overset{\simeq}{\to} \pi^{-1}(U_\alpha)$, with U_α connected. Fix an orientation of F. The connectedness assumption implies that, for each α, the linear maps $\psi_{\alpha,x} : F \to F_x, x \in U_\alpha$ either all preserve, or all reverse orientations. Let ρ be an orientation reversing isomorphism of F. Define a new trivialization (U_α, ϕ_α) by:

$$\phi_\alpha(x,v) = \begin{cases} \psi_\alpha(x,v) & \text{if} \psi_{\alpha,x} \text{preserves orientations} \\ \psi_\alpha(x,\rho(v)) & \text{if} \psi_{\alpha,x} \text{reverses orientations} \end{cases}$$

Then each $\phi_{\alpha,x}$ preserves orientations. Hence so does $\phi_{\alpha,x}^{-1} \circ \phi_{\beta,x}$, i.e. $\det(\phi_{\alpha,x}^{-1} \circ \phi_{\beta,x}) > 0$.

The converse is also true, but we will only need the implication we proved.

We now proceed to define the vertical bundle. Given a smooth fiber bundle $F \to E \overset{\pi}{\to} B$ and a point $z \in E$, the vertical subspace of $T_z(E)$ is defined to be $V_z(E) := ker(d\pi)_z$. Surjectivity of $(d\pi)_z$ implies $\dim V_z(E) = \dim E - \dim B = \dim F$. For $a \in B, z \in F_a, V_z(E) = im(dj_a)_z$, where $j_a : F_a \to E$ is the standard inclusion. Indeed, $\pi \circ j_a$ is constant, hence $d\pi \circ dj_a = 0$ and this implies $im(dj_a)_z \subset V_z(E)$. Injectivity of $(dj_a)_z$ implies $\dim(im(dj_a)_z) = \dim F = \dim V_z(E)$ and hence the equality.

The vertical subbundle is defined to be $V_E := \cup_{z \in E} V_z(E)$. It is a subbundle of τ_E. This follows easily by choosing a trivialization (U_α, ψ_α) and considering diagram (2), as a restriction of diagram (1).

$$\begin{tikzcd} \tau_{U_\alpha} \times \tau_F \arrow[r, "d\psi_\alpha \simeq"] \arrow[d] & \tau_{\pi^{-1}(U_\alpha)} \arrow[d] \\ U_\alpha \times F \arrow[r, "\psi_\alpha \simeq"] & \pi^{-1}(U_\alpha) \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} U_\alpha \times \tau_F \arrow[r, "\simeq"] \arrow[d] & V_{E/\pi^{-1}(U_\alpha)} \arrow[d] \\ U_\alpha \times F \arrow[r, "\psi_\alpha \simeq"] & \pi^{-1}(U_\alpha) \end{tikzcd} \tag{2}$$

The equality $V_z(E) = im(dj_a)_z$ implies that the derivative $dj_a : \tau_{F_a} \to \tau_E$ can be considered as a bundle map $dj_a : \tau_{F_a} \to V_E$ inducing linear isomorphisms on the fibres.

We now return to the setting of the previous section. Denoting by $S(\tau_M \oplus 1)$ and V_E the total space of the sphere bundle of M^n and the vertical subbundle, respectively, we wish to construct a map $V_E \overset{\alpha}{\to} \tau_M$ making the diagram

$$\begin{tikzcd}
V_E \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \tau_M \arrow[d, "\pi"] \\
S(\tau_M \oplus 1) \arrow[r, "\pi"] & M
\end{tikzcd}$$

commutative and inducing linear isomorphisms on the fibres.

Let $F_x \simeq S^n$ be a fiber of the sphere bundle of $M, x \in M$. For $z \in F_x$, we know that $(dj_x)_z^{-1} : V_z(E) \to T_z(F_x)$ is a linear isomorphism. The map $(\tau_M)_x \to F_x$ given by the inverse of stereographic projection has injective derivative and equality of dimensions shows that it induces a linear isomorphism of tangent spaces. Identifying $(\tau_M)_x$ with its tangent space and composing these three maps, gives the desired linear isomorphism.

We will need the map α for the proofs in the next section.

We conclude this section by showing that the orientation class of E, the total space of $S(\tau_M \oplus 1)$, induces an orientation of V_E.

Note that there is a map $\rho_V : \Omega^n(E) \longrightarrow Sec(\wedge^n V_E^*)$ given by $\rho_V \Phi_z(\xi_1 \dots \xi_n) = \Phi_z(\xi_1 \dots \xi_n), z \in E, \xi_i \in V_z(E)$. We can regard $\wedge(dj_x)_z, x = \pi z$ as an isomorphism $\sigma_z : \wedge^n T_z(F_x) \xrightarrow{\simeq} \wedge^n(V_z(E))$. Then for $\Psi \in \Omega^n(E), \sigma_z^*(\rho_V \Psi_z) = (j_x^* \Psi)_z, z \in E, x = \pi z$. If in addition Ψ orients the sphere bundle, $(j_x^* \Psi)_z \neq 0$, for every $z \in F_x, x \in M$. Hence $(\rho_V \Psi)_z \neq 0, z \in E$. By **Lemma 5**, we get that the vector bundle $V_E \to E$ is orientable.

4.3 An Explicit Orientation Class and its Relation to the Euler Class

The considerations in the previous section show that we can consider the following closed n-form on E: $u = \frac{1}{2}[Pf(-\frac{F^\bigtriangledown}{2\pi}) - \pi^* s_\infty^* Pf(-\frac{F^\bigtriangledown}{2\pi})]$, where $E \xrightarrow{\pi} M$, and $F^\bigtriangledown$ is the curvature of some metric connection on E.

We will prove in this section that u is an orientation class and $s_0^*(u)$ is the Euler class of τ_M.

We first show that given a vector bundle $\tilde{E} \xrightarrow{\tilde{\pi}} \tilde{B}$ with a Riemannian metric $\tilde{g}$ and a compatible metric connection $\tilde{\bigtriangledown}$, a commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\
B \arrow[r, "\Psi"'] & \tilde{B}
\end{tikzcd}$$

with φ restricting to linear isomorphisms on the fibers, we can pull back $\tilde{g}$ and $\tilde{\bigtriangledown}$ to g and $\bigtriangledown$, respectively. Moreover, $\bigtriangledown$ is compatible with the metric g and the connection matrix of 1-forms for $\bigtriangledown$ is obtained by pullback under Ψ.

Note that there is a map $\varphi^* : Sec(\tilde{E}) \longrightarrow Sec(E)$ given by $(\varphi^*(\tilde{s}))(x) = \varphi_x^{-1}[s(\Psi(x))], x \in B$. This extends to a map $\varphi^* : \Omega^p(\tilde{B}, \tilde{E}) \longrightarrow \Omega^p(B, E)$ by: $(\varphi^* \Omega)_x(X_1 \dots X_p) = \varphi_x^{-1}[\Omega_{\Psi(x)}((d\Psi)X_1 \dots (d\Psi)X_p)]$. For $a \in \Omega^*(B)$, $b \in \Omega^*(B, E)$, $\varphi^*(a \wedge b) = \Psi^* a \wedge \varphi^* b$. Given a connection $\bigtriangledown$ on E, φ will be called connection-preserving if $\bar{\bigtriangledown}\varphi^* = \varphi^* \tilde{\bigtriangledown}$. We now prove:

Lemma 6 Given a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \tilde{E} \arrow[d, "\tilde{\pi}"] \\ B \arrow[r, "\Psi"'] & \tilde{B} \end{tikzcd}$$

with φ restricting to linear isomorphisms on the fibers, to any connection $\tilde{\bigtriangledown}$ on $\tilde{E}$ corresponds a unique connection $\bigtriangledown$ on E, such that φ is connection preserving. (This is, by definition, the pullback of $\tilde{\bigtriangledown}$.)

Proof There is an isomorphism $\Omega(B) \otimes_{\Omega^0(\tilde{B})} Sec(\tilde{E}) \simeq \Omega(B, E)$ given by $\Phi \otimes s \longrightarrow \Phi \wedge \varphi^* s$. (It can be obtained by using $\Omega(B) \otimes_{\Omega^0(B)} Sec(E) \simeq \Omega(B, E)$ and $\Omega(B) \otimes_{\Omega^0(\tilde{B})} Sec(\tilde{E}) \simeq Sec(E)$; the latter is given by $f \otimes s \leftrightarrow f(\varphi^* s)$ and in particular shows that $Sec(E)$ is generated by sections of the form $\varphi^* s$, $s \in Sec(\tilde{(E)})$, so it suffices to know how $\bigtriangledown$ acts on these sections.)

To construct $\bigtriangledown$, define an $\mathbf{R}$-bilinear map: $\beta : \Omega^0(B) \times Sec(\tilde{(E)}) \longrightarrow \Omega^1(B, E)$ by

$$\beta(f, \sigma) = df \wedge \varphi^* \sigma + f \cdot \varphi^*(\tilde{\bigtriangledown} \sigma).$$

Then $\beta(f \cdot \tilde{g}, \sigma) = \beta(f, \tilde{g} \cdot \sigma)$, $\tilde{g} \in \Omega^0(\tilde{B})$, where the dots denote the $\Omega^0(\tilde{B})$-module multiplication, hence β descends to an $\mathbf{R}$-linear map $\bigtriangledown : \Omega(B) \otimes_{\Omega^0(\tilde{B})} Sec(\tilde{E}) \longrightarrow \Omega^1(B, E)$; identifying $\Omega(B) \otimes_{\Omega^0(\tilde{B})} Sec(\tilde{E})$ with $Sec(E)$, we get the desired connection.

We now check that $\bigtriangledown$ preserves the pullback metric g. Note that with this metric φ induces an isometry on the fibres. By the remarks in the proof of **Lemma 6**, it suffices to check metric-compatibility for sections that are pullbacks of sections of $\tilde{E} \xrightarrow{\tilde{\pi}} \tilde{B}$. We have:

$$g(\bigtriangledown \varphi^* \tilde{s_1}, \varphi^* \tilde{s_2}) + g(\bigtriangledown \varphi^* \tilde{s_2}, \varphi^* \tilde{s_1}) = g(\varphi^* \tilde{\bigtriangledown} \tilde{s_1}, \varphi^* \tilde{s_2}) + g(\varphi^* \tilde{\bigtriangledown} \tilde{s_2}, \varphi^* \tilde{s_1}) \overset{since \varphi isometry}{=}$$

$$\tilde{g}(\tilde{\bigtriangledown} \tilde{s_1}, \tilde{s_2}) + \tilde{g}(\tilde{\bigtriangledown} \tilde{s_2}, \tilde{s_1}) = d\tilde{g}(\tilde{s_1}, \tilde{s_2}) \overset{since \varphi isometry}{=} dg(\varphi^* \tilde{s_1}, \varphi^* \tilde{s_2}).$$

Last, we note that the connection matrix of 1-forms for $\bigtriangledown$ is obtained by pullback under Ψ. It suffices to note that if $\tilde{s_j}$ are local trivializing sections , then so are $\varphi^* \tilde{s_j}$ and if $\tilde{\bigtriangledown} \tilde{s_i} = \sum \tilde{\omega_{ij}} \otimes \tilde{s_j}$ then $\bigtriangledown \varphi^* \tilde{s_i} = \varphi^*(\sum \tilde{\omega_{ij}} \otimes \tilde{s_j}) = \Psi^* \tilde{\omega_{ij}} \otimes \varphi^* \tilde{s_j}$.

The preceding paragraph implies that the curvature matrix for $\bigtriangledown$ is obtained from that of $\tilde{\bigtriangledown}$ by pullback under Ψ; if in addition the bundles are oriented , this implies $\Psi^* Pf(F^{\tilde{\bigtriangledown}}) = Pf(F^{\bigtriangledown})$.

We apply the considerations sofar to conclude that u is an orientation class. In the commutative diagram

$$\begin{tikzcd} \tau_{S^n} \arrow[r, "dj_x"] \arrow[d] & V_E \arrow[d] \\ S^n \arrow[r, hook, "j_x"] & E \end{tikzcd}$$

, dj_x induces linear isomorphisms on the fibres, hence for an arbitrary metric and compatible metric connection on $V_E \longrightarrow E$, $\frac{1}{2} \int_{S^n} j_x^* Pf(-\frac{F^{\bigtriangledown}}{2\pi}) = 1$, by the proof for the case of spheres. To show that $\int_{S^n} j_x^* \pi^* s_\infty^* Pf(-\frac{F^{\bigtriangledown}}{2\pi}) = 0$, we note that $j_x^* \pi^*$ is the zero map in cohomology , so it maps any form to a closed form. The conclusion follows by Stokes' theorem.

Finally, we conclude that $s_0^*(u)$ represents the Euler class of τ_M. We use the commutative diagram

$$\begin{tikzcd} V_E \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \tau_M \arrow[d, "\pi"] \\ S(\tau_M \oplus 1) \arrow[r, "\pi"] & M \end{tikzcd}$$

from section **4.2**. Given a connection ∇ on $\tau_M \longrightarrow M$, we have $\pi^* Pf(-\frac{F^{\nabla}}{2\pi}) = Pf(-\frac{F^{(\alpha^*\nabla)}}{2\pi})$, hence $Pf(-\frac{F^{\nabla}}{2\pi}) = s_0^* Pf(-\frac{F^{(\alpha^*\nabla)}}{2\pi})$.

$s_0^*(u) = \frac{1}{2}(s_0^* - s_\infty^*) Pf(-\frac{F^{(\alpha^*\nabla)}}{2\pi}) \overset{(s_0+s_\infty=0)}{=} s_0^* Pf(-\frac{F^{(\alpha^*\nabla)}}{2\pi}) = Pf(-\frac{F^{\nabla}}{2\pi})$, q.e.d.

4.4 Final Step of the Proof: $\int_M s_0^*(u) = \chi(M)$

Let M be any compact oriented manifold. Let $\{\omega_i\}$ be a basis of $H^*(M)$ and $\{\tau_j\}$ the dual basis under Poincar*é* duality, i.e., $\int_M \omega_i \wedge \tau_j = \delta_{ij}$. Denote by π and ρ the projections of $M \times M$ onto the first and second factor, respectively. By the K*ü*nneth isomorphism, $H^*(M \times M) = H^*(M) \otimes H^*(M)$, a basis of the former being given by $\{\pi^*\omega_i \wedge \rho^*\tau_j\}$. Let $\eta_\Delta = \sum c_{ij} \pi^*\omega_i \wedge \rho^*\tau_j$ be the Poincar*é* dual of the diagonal Δ in $M \times M$. We recall that given an oriented manifold M^n and a compact oriented submanifold S^k, the Poincar*é* dual of S is the unique class $[\eta_S] \in H_c^{n-k}(M)$ such that $\int_S i^*\omega = \int_M \omega \wedge \eta_S$, for any closed form $\omega \in \Omega^k(M)$.

We prove the following lemma:

Lemma 7 $\eta_\Delta = \sum (-1)^{deg\omega_i} \pi^*\omega_i \wedge \rho^*\tau_j$ and $\int_\Delta \eta_\Delta = \chi(M)$.

Proof We compute $\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l$ in two ways. On one hand we can pull back this integral back to M via the diagonal map $i : M \longrightarrow \Delta \subset M \times M$:

$$\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l = \int_M i^*\pi^*\tau_k \wedge i^*\rho^*\omega_l = \int_M \tau_k \wedge \omega_l = (-1)^{(deg\tau_k)(deg\omega_l)} \delta_{kl}.$$

On the other hand, by the definition of the Poincar*é* dual of a submanifold,

$$\int_\Delta \pi^*\tau_k \wedge \rho^*\omega_l = \int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \eta_\Delta = \sum_{i,j} c_{ij} \int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \pi^*\omega_i \wedge \rho^*\tau_j$$

$$= \sum_{i,j} c_{ij} (-1)^{(deg\tau_k + deg\omega_l)(deg\omega_i)} \int_{M\times M} \pi^*(\omega_i \wedge \tau_k) \rho^*(\omega_l \wedge \tau_j) = (-1)^{(deg\tau_k + deg\omega_l) deg\omega_k} c_{kl}.$$

Hence $c_{kl} = \begin{cases} 0 & \text{if} k \neq l \\ (-1)^{deg\omega_k} & \text{if} k = l \end{cases}$

$$\int_{\Delta} \eta_{\Delta} = \sum_i (-1)^{deg\omega_i} \int_{\Delta} \pi^* \omega_i \wedge \rho^* \tau_i = \sum_i (-1)^{deg\omega_i} \int_M i^* \pi^* \omega_i \wedge i^* \rho^* \tau_i =$$

$$\sum_i (-1)^{deg\omega_i} \int_M \omega_i \wedge \tau_i = \sum_i (-1)^{deg\omega_i} = \sum_q (-1)^q dim H^q(M) = \chi(M), \text{q.e.d.}$$

A result we will use without proof is the following lemma, a particular case of the integration across the fiber homomorphism.

Lemma 8 Let $E \xrightarrow{\pi} M^n$ be an oriented fiber bundle with compact fibers over an oriented manifold. Let u be an orientation class. Then $\int_M \omega = \int_E \pi^* \omega \wedge u$, for any $\omega \in \Omega_c^n(M)$. Here E carries the canonical orientation induced by an oriented trivializing atlas of M.

In our context, if $\omega = s_0^* \tau$, then $\int_M \omega = \int_{s_0(M)} \tau = \int_E \pi^* s_0^* \tau \wedge u$. But $s_0 \circ \pi$ is homotopic to id_E, since the fibers, being even spheres, are simply-connected. Hence $\int_E \pi^* s_0^* \tau \wedge u = \int_E \tau \wedge u$ and the Poincaré dual of the zero section can be represented by the orientation class.

The map $(p, v) \xrightarrow{f} (p, exp_p v)$ induces an orientation preserving diffeomorphism of an open neighborhood O of the zero section in E onto an open neighborhood D of the diagonal Δ in $M \times M$. Here *exp* denotes the exponential map and v is obtained from an element in the fiber at p by omitting the last coordinate. It is straightforward to check that the integral of the Poincaré dual of $s_0(M)$ in O over $s_0(M)$ equals the integral of the Poincaré dual of Δ in D over Δ. The localization principle (the support of the Poincaré dual of a compact oriented submanifold S may be shrunk into any open neighborhood of S), together with the earlier considerations gives:

$$\chi(M) = \int_{\Delta} P.D._{M \times M}(\Delta) = \int_{s_0(M)} P.D._E(s_0(M)) = \int_{s_0(M)} u = \int_M s_0^*(u).$$

With these, the proof of the Gauss-Bonnet theorem is complete.

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