

Some Results on Cyclable Graphs

Irina Goia

Department of Mathematics, MIT, Cambridge, MA 02139 *

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Abstract

In this paper we improve some of the results on cyclable graphs achieved by Klostermeyer and Soltes in [1]. A simple graph G is defined to be *cyclable* if for each orientation D of G there is a set S of the vertices, such that reversing all the arcs of D with one end in S , results in a hamiltonian digraph. We show that any graph with minimal degree $n/2 + 1$ (n even), with two exceptions, is cyclable. We also show that the square of any 3-connected graph is cyclable and that the third power of a connected graph which contains a vertex of degree greater or equal to 3, is cyclable.

1 Introduction

In order to define cyclable graphs, Klostermeyer and Soltes [1] introduced a transformation on the vertices of an oriented graph. When a vertex of an oriented graph is *pushed*, the orientation of each of its incident arcs is reversed. A simple graph is said to be *cyclable* if for each orientation of its edges there is a subset of vertices that can be pushed, such that the resulting digraph has an oriented hamiltonian cycle. Note that the order in which the vertices of a set are pushed does not matter.

A motivation for studying cyclable graphs is the fact that known conditions that guarantee hamiltonicity often imply stronger results about the graph such as pancyclicity [7] and cycle-extendability [4]. In [3], Klostermeyer and Soltes show the following:

Theorem 1 *For n odd, any hamiltonian graph on n vertices is cyclable. For n even, an orientation of a simple graph is pushable to one that contains a directed hamiltonian cycle if and only if it contains an oriented cycle passing through all the vertices with an even number of arrows clockwise.*

*Send all correspondence to: Irina Goia, 70 Amherst St., 309, 02142, Cambridge, MA; irina@mit.edu

In [1], Klostermeyer and Soltes study the cyclability property in powers of graphs. They show that the fourth power of any connected graph is cyclable, that the third power of any 2-connected graph is cyclable, and ask whether the square of a 3-connected graph is cyclable. We give a simple proof of this conjecture. Moreover, we strengthen the previous results by proving that the third power of any connected graph which contains a vertex of degree greater than 2, is cyclable.

Results concerning hamiltonicity in powers of graphs were obtained in the 60s by Sekanina and Karaganis and in the 70s by Fleischner. Sekanina [5] and Karaganis [6] proved that the cube of any connected graph is hamiltonian; Fleischner [9], [10] proved that the square of any 2-connected graph is hamiltonian. These results motivated stronger ones: in the same years, it was proved that the same conditions imply hamiltonian-connectedness [11] and other properties of the graph [7], [8]. We will use these results in our proofs.

In [1], Klostermeyer and Soltes also obtained a sufficient condition on the minimal degree of a graph, that implies cyclability. They showed that any graph on n vertices with minimal degree $n/2 + 2$ is cyclable. If n is even (we assume this, since for n odd, cyclability is equivalent to hamiltonicity), we show that a stronger result holds: any graph with minimal degree $n/2 + 1$, except for K_4 and $K_{2,2,2}$, is cyclable.

2 Preliminary Lemmas

In our proofs, we are going to pay a special attention to even length cycles, in particular of length 4 and n (the number of vertices of the graph). A cycle of even length of an oriented graph is said to be *good* if it has an even number of arrows in one direction, as we go across the cycle. Otherwise, it is said to be *bad*. If a vertex is pushed, good cycles remain good and bad cycles remain bad. According to Theorem 1, a graph on n vertices (n even) is cyclable if and only if, in each of its orientations we can find a good n -cycle.

Lemma 1 *If two vertices of a graph, x and y , have three common neighbors labeled 1,2,3, then one of the cycles $x1y2$, $x1y3$, $x2y3$, is good.*

Proof. Assume $x1y2$ and $x1y3$ are bad cycles. An edge $a_k a_{k+1}$ of a directed cycle $a_1 a_2 \dots a_m$ (m even and not necessarily all edges being oriented clockwise) is said to be good if it is oriented from a_k to a_{k+1} ; otherwise it is said to be bad. Note that if we go across the cycle in one direction and then in the opposite direction, all edges that were good the first time, are bad the second time. Because we are interested in the parity of good or bad edges across the cycle and m is even, it does not matter in what direction we go across the cycle.

The assumption implies that each of the sets $\{x1, 1y, y2, 2x\}$ and $\{x1, 1y, y3, 3x\}$ contains an odd number of bad edges. Their "union" $\{x1, x1, 1y, 1y, y2, y3, 2x, 3x\}$

will therefore contain an even number of bad edges. But $\{x1, x1, 1y, 1y\}$ contains an even number of bad edges. So $\{y2, y3, 2x, 3x\}$ must also contain an even number of bad edges. If we “look oppositely” at an even number of edges in any set, the parity of the number of bad edges does not change. So $\{2x, x3, 3y, y2\}$ contains an even number of bad edges and therefore is good.

Lemma 2 *If 1234 is a good 4-cycle of an orientation of a simple graph with n vertices (n even) and the rest of the vertices can be partitioned in two classes, v_1, \dots, v_k and w_1, \dots, w_l , such that $v_1 \dots v_k$ is a path from 1 to 3 and $w_1 \dots w_l$ is a path from 2 to 4 and k and l are even, then that orientation is pushable to a hamiltonian one. (In particular if 13 is an edge and 2 and 4 are hamiltonian-connected in $G - \{1, 3\}$, then, no matter what the orientation is, it is pushable to a hamiltonian one.)*

Proof. Denote by p_1 and p_2 the paths from 1 to 3 and from 4 to 2, respectively. Assume the given orientation is not pushable to a hamiltonian one. Then, by Theorem 1, both n -cycles $1p_134p_221$ and $1p_132\overline{p_2}41$ are bad ($\overline{p_2}$ is the path p_2 in reverse order). So, the number of bad edges in their “union” $\{p_1, p_1, 34, p_2, \overline{p_2}, 21, 32, 41\}$ is even. The number of bad edges in $\{p_1, p_1\}$ is even and the number of bad edges in $\{p_2, \overline{p_2}\}$ equals the length of p_2 , which is odd. So, in the set $\{34, 21, 32, 41\}$ there must be an odd number of bad edges; the same holds for the set $\{34, 12, 23, 41\}$. We get that the 4-cycle 1234 is bad, a contradiction.

Corollary 1 *If a simple graph G contains 2 adjacent vertices x and y , which have three common neighbors labeled 1, 2, 3 and the pairs of vertices $1 - 2, 1 - 3, 2 - 3$ are hamiltonian-connected in $G - \{x, y\}$, then G is cyclable.*

Using the same techniques, we can show that if an orientation D of a simple graph G with n vertices (n even) contains a bad 4-cycle 1234 and the rest of the vertices can be partitioned in two classes v_1, \dots, v_k and w_1, \dots, w_l , such that $v_1 \dots v_k$ are the vertices of a path between 1 and 3, while $w_1 \dots w_l$ are the vertices of a path between 4 and 2, with k and l odd, then D is pushable to a hamiltonian orientation. We can also show that if a graph G with n vertices (n even) contains a 4-cycle for which the rest of the vertices can be partitioned as above, once for k, l odd and once for k, l even, then G is cyclable.

3 Proof of the Results

3.1 Degree Results

Theorem 2 *Any graph with n vertices and with minimal degree $n/2 + 1$ (n even) is cyclable, except for K_4 and $K_{2,2,2}$.*

Proof. For any two adjacent vertices, v and w , $d_{G-\{w\}}(v) + d_{G-\{v\}}(w) \geq n$ and because $|N_{G-\{w\}}(v) \cup N_{G-\{v\}}(w)| \leq n - 2$, v and w have at least two common neighbors in $G - \{w, v\}$. Therefore, G will contain 4-cycles with a diagonal. Let D be an arbitrary orientation of G . We consider two cases.

Case 1: D contains a good 4-cycle with a diagonal, say 1234. For each $v, w \in G - \{3\}$, $d_{G-\{3\}}(v) + d_{G-\{3\}}(w) \geq n$, so, by a result of Ore, v and w are hamiltonian-connected in $G - \{3\}$, or otherwise said, there is a hamiltonian cycle in G in which 3 is adjacent to 2 and 4. Let the vertices of the cycle be: $4, 3, 2, v_1, \dots, v_s = x, 1, v_{s+1} = y, v_{s+2}, \dots, v_{n-4}$.

Subcase a: 1 is adjacent to 2 or 4 on the cycle. Without loss of generality, assume it is adjacent to 2. If we show that 2 and 4 are hamiltonian connected in $G - \{1, 3\}$, by Lemma 2, we are done.

Let the vertices of the cycle be $4, 3, 2, 1, v_1, v_2, \dots, v_{n-4}$ in this order. Assume 2 and v_1 are nonadjacent, otherwise $2v_1 \dots v_{n-4}4$ is the desired path. Let k be the number of neighbors of 2 among v_2, \dots, v_{n-4} ; then $k \geq n/2 - 2$, otherwise $d_G(2) \leq n/2$. Now if 2 is adjacent to v_j , $j \in \{2, \dots, n-4\}$ and v_1 is adjacent to v_{j+1} ($v_{n-3} = 4$) we are again done since $2v_jv_{j-1} \dots v_1v_{j+1} \dots 4$ is a hamiltonian path between 2 and 4 in $G - \{1, 3\}$. Therefore, we may assume that if 2 is adjacent to v_j , v_1 is not adjacent to v_{j+1} . Then v_1 is going to have at most $n - 5 - k \leq n - 5 - n/2 + 2 = n/2 - 3$ neighbors among v_3, \dots, v_{n-4} , 4 and $d_G(v_1) \leq n/2 - 3 + 3 = n/2$, a contradiction.

Subcase b: Neither 2, nor 4, are neighbors of 1 on the cycle. Let the vertices of the cycle be, in this order $4, 3, 2, v_1, \dots, v_s = x, 1, y = v_{s+1}, \dots, v_{n-4}, 4$. If x and y are adjacent, $2v_1 \dots xyv_{s+2} \dots 4$ is the desired path. Assume x and y are nonadjacent. Let k be the number of neighbors of x among $v_{s+2}, \dots, v_{n-4}, 4$ and l be the number of neighbors of x among $2, v_1, \dots, v_{s-1}$. Then $d_G(x) \geq n/2 + 1$ implies $k + l \geq n/2 - 1$.

If x is adjacent to v_j , $j \in \{s+2, \dots, n-4\}$, v_{j+1} ($v_{n-3} = 4$) is “interdicted” for y , in the sense that if v_{j+1} were adjacent with y , the path $2v_1 \dots v_{s-1}xv_jv_{j-1} \dots yv_{j+1} \dots 4$ would be a hamiltonian path between 2 and 4. Therefore, among the vertices $v_{s+2}, \dots, v_{n-4}, 4$, there are $k - 1$ or k vertices “interdicted” for y (depending on whether x is adjacent with 4 or not).

If x is adjacent to v_j , $j \in \{0, \dots, s-2\}$ ($v_0 = 2$), then v_{j+1} is “interdicted” for y ; namely, if y were adjacent to v_{j+1} , then $2v_1 \dots v_jxv_{s-1} \dots v_{j+1}yv_{s+2} \dots v_{n-4}4$ is a hamiltonian path between 2 and 4 in $G - \{1, 3\}$. Therefore there are $l - 1$ “interdicted” vertices for y among v_1, \dots, v_{s-1} .

Now y is adjacent to 1, may be adjacent to 2 and 3, and is not adjacent to x ; among the remaining vertices, y may be adjacent only to noninterdicted vertices. We get that

$$\begin{aligned} d_G(y) &\leq 3 + (s - 1 - (l - 1)) + (n - 4 - s - (k - 1)) = \\ &n - l - k \leq n - (n/2 - 1) = n/2 + 1. \end{aligned}$$

From the hypothesis, $d_G(y) \geq n/2 + 1$, so equality must hold. This implies that y is adjacent to 2 and 3, while x is adjacent to 4. Because the configuration is symmetric in x and y , by using similar arguments, we also get that x is adjacent to 3.

In the case that G has only 6 vertices, 1, 2, 3, 4, x , y , because the degree of each vertex is at least 4, and x and y are nonadjacent, x must be adjacent to 2 and y to 4. The resulting graph is cyclable by Corollary 1. (Since 2 and 3 have x , 1 and y as common neighbors; any two of the latter are hamiltonian-connected in $G - \{2, 3\}$.)

Assume G has at least 8 vertices. Let $v_1 = \beta$ and $v_{n-4} = \alpha$. We again consider two subcases:

Subcase a': x is not adjacent to 2 and y is not adjacent to 4 on the initial cycle. If we look at the paths $4v_s v_{s-1} \dots v_2 v_1$ and $2v_{s+1} v_{s+2} \dots v_{n-4}$, we notice that they play the roles that the paths $4v_{n-4} \dots v_{s+1}$ and $2v_1 v_2 \dots v_s$, respectively, played in the previous proof (v_1 plays the role of y and v_{n-4} the role of x). Again following arguments about “interdicted” vertices, we get that both v_1 and v_{n-4} must be adjacent to 1 and 3 (assuming that 2 and 4 are not hamiltonian-connected in $G - \{1, 3\}$).

In the resulting configuration 1 and 2 have as common neighbors the vertices labeled β , 3 and y . We will show that any two of the later are hamiltonian-connected in $G - \{1, 2\}$, so D is pushable to a hamiltonian orientation, by Corollary 1. Indeed, $3v_1 v_2 \dots v_s 4v_{n-4} \dots v_{s+2} y$, $\beta v_2 \dots v_s 3 4v_{n-4} \dots v_{s+2} y$ and $\beta v_2 \dots v_s 4v_{n-4} \dots v_{s+2} y 3$ are the desired hamiltonian paths in $G - \{1, 2\}$.

Subcase b': Either x is adjacent to 2 or y is adjacent to 4 on the initial cycle. Assume the former takes place. Denote by $w_1 \dots w_{n-6}$ the path in the initial cycle between y and 4. Again we can apply Corollary 1: x , 1 and y are adjacent to 2 and 3; in $G - \{2, 3\}$, $4x1yw_1 \dots w_{n-6}$ is a hamiltonian cycle, so x and 1, y and 1 are hamiltonian-connected. Moreover, $yw_1 \dots w_{n-6} 41x$ is a hamiltonian path between x and y in $G - \{2, 3\}$.

Case 2: Every 4-cycle with a diagonal in G is bad under D . Then any two adjacent vertices, v and w , have exactly two common neighbors. Indeed, if they had more than two, there would be a good 4-cycle, by Lemma 1; if they had less than two, one of them would have degree less than $n/2 + 1$. Moreover, any vertex in $G - \{v, w\}$ is adjacent to at least one of v and w and G is $n/2 + 1$ regular. Let x and y be adjacent vertices and a and b their common neighbors. We have two subcases:

Subcase a: a and b are not adjacent; then x and a have only one other common neighbor besides y ; call it c . Then y is not adjacent to c (otherwise, x and y would have at least three common neighbors). Also, a and y have exactly one other neighbor beside x . Call it d ; then x and d are nonadjacent. If G had another vertex v beside a, b, c, d, x, y , then v would necessarily be adjacent to two of a, x, y . These two neighbors of v will have at least three common neighbors in G , contradiction. So G is a graph on 6 vertices; because $d_G(b) \geq 4$

and b is not adjacent to a , it must be adjacent to c and d . We easily get that G is $K_{2,2,2}$, the complete tripartite graph on 6 vertices, which can be checked by hand that is not cyclable.

Subcase b: a and b are adjacent. So G contains K_4 . If G had a fifth vertex w , w must be adjacent to two of a, b, x and y , by the argument given at the end of Case 2 above. These two neighbors of w would have three neighbors in G , contradiction.

With this, the proof is complete.

When $n = 4t + 2$, the bound $n/2 + 1$ in Theorem 2 is sharp. Indeed, for $n = 4t + 2$, any graph H such that $\overline{K_{2t+1}} + \overline{K_{2t+1}} \subseteq H \subseteq K_{2t+1} + \overline{K_{2t+1}}$, (where $G_1 + G_2$ is obtained by joining each vertex of G_1 with each vertex of G_2) is not cyclable. Let the vertices of H be $w_1, \dots, w_{2t+1}, v_1, \dots, v_{2t+1}$, where w_1, \dots, w_{2t+1} are independent, $v_i w_j$ is an edge for all i and all j , and $v_i v_j$ may or may not be an edge for all i and all j .

We will prove that any hamiltonian cycle in H is also a hamiltonian cycle in the complete bipartite graph with partite classes $\{v_1, \dots, v_{2t+1}\}$ and $\{w_1, \dots, w_{2t+1}\}$. Indeed, if we denote by v_{i_s} and v'_{i_s} the neighbors of w_s on the hamiltonian cycle, the set $V = \{v_{i_1}, \dots, v_{i_{2t+1}}, v'_{i_1}, \dots, v'_{i_{2t+1}}\}$ contains only elements from $\{v_1, \dots, v_{2t+1}\}$, since w_1, \dots, w_{2t+1} are independent and each v_i appears in V at most twice. This implies that each v_i appears in V exactly twice, or, otherwise said, no edge $v_i v_j$ lies on the hamiltonian cycle.

Because of the above fact and because $K_{r,r}$ is cyclable if and only if r is even (Klostermeyer and Soltes, [1]), we get that H can not be cyclable.

It would be of interest to find a family of graphs with minimal degree $n/2$, $n = 4t$, that are not cyclable, or even more, to characterize all graphs with minimal degree $n/2$ that are not cyclable.

Theorem 2 and known results on hamiltonicity prompt us to make the following conjecture.

Conjecture 1 *a) Any graph with n vertices such that the sum of the degrees of any two nonadjacent vertices is greater or equal to $n + 2$ is cyclable (except perhaps for some particular cases).*

b) If $d_1 \leq \dots \leq d_n$ is the degree sequence of a graph with n vertices and $d_{k-1} \leq k$ together with $2 \leq k \leq n/2$, implies $d_{n-k} \geq n - k + 2$, then the graph is cyclable. (except perhaps for some particular cases).

3.2 Results for Powers of Graphs

Theorem 3 *The square of any 3-connected graph is cyclable.*

Proof. We will use the following theorem due to Tutte [2], which describes how every 3-connected graph can be obtained from a wheel.

Theorem 4 *A graph is 3-connected if and only if it is a wheel or can be obtained from a wheel by repeated applications of the following two operations.*

1. *The addition of an edge.*
2. *The replacement of a vertex x of degree greater or equal to 4 by two adjacent vertices x', x'' and joining every neighbor of x to exactly one of x', x'' in such a way that both x' and x'' will have degree greater or equal to 3. (This operation is called a splitting of the vertex x .)*

Let G be a 3-connected graph obtained by the above procedure and let v_1 and v_2 be the vertices obtained after the last splitting. (If there was no splitting, then G is a wheel, and its square is a complete graph, which is cyclable) Let x_1, \dots, x_k be the neighbors of v_1 and y_1, \dots, y_l the neighbors of v_2 , with $k, l \geq 2$.

Let H be the 3-connected graph from which G was obtained and let v be the vertex of H which was split to produce v_1 and v_2 . Then $H - \{v\}$, which is the same as $G - \{v_1, v_2\}$, is 2-connected, so its square is hamiltonian-connected. In particular, any two of x_1, x_2 and y_1 are hamiltonian-connected in the square of $G - \{v_1, v_2\}$. Also, in the square of G , v_1 and v_2 have as common neighbors x_1, x_2 and y_1 . By applying Corollary 1, we get that the square of G is cyclable.

This completes the proof of Theorem 3.

Theorem 5 *The cube of a connected graph that has a vertex of degree greater or equal to 3, is cyclable.*

Proof. Without loss of generality, we assume that the graph is minimally connected (it is a tree), and has an even number of vertices. Let a be a vertex of degree greater than two and let 3 and 1 be two arbitrarily chosen neighbors of a . We may assume that 3 has another neighbor labeled 2.

In the cube of G , we look at the cycle $a321$. We are going to show that there is a path $av_1 \dots v_s 2$, such that $G - \{v_1, \dots, v_s, a, 1, 2, 3\}$ are the vertices of a path between 1 and 3. If we delete the edge $a3$ from G , we get two trees: T_1 , which contains 3, and T_2 .

Let $t_1, \dots, t_p, 2 = t_{p+1}$ be the neighbors of 3 in T_1 . Let $t'_1, \dots, t'_p, t'_{p+1}$ be neighbors of t_1, \dots, t_p, t_{p+1} , respectively (if they exist). Let $T'_1, \dots, T'_p, T'_{p+1}$ be the trees obtained from T_1 by deleting, respectively, the edges $3t_1, \dots, 3t_p, 3t_{p+1}$. (T_j contains t_j .) Then, for all i , the cube of T_i contains a hamiltonian path P_i between t_i and t'_i . The desired path between a and 2 in the cube of G is:

$$at'_1 P_1 t_1 t'_2 P_2 t_2 \dots t'_p P_p t_p t'_{p+1} P_{p+1} t_{p+1},$$

where $t_{p+1} = 2$. (Notice that if some t'_i do not exist, we still get the desired path.)

Now we are going to construct a path with the property mentioned at the beginning between 1 and 3. Let a_1, \dots, a_r be all the neighbors of a besides 1 and 3. Let R_1, \dots, R_r be the trees obtained from G by deleting, respectively, the edges aa_1, \dots, aa_r (R_j contains a_j). Let a'_1, \dots, a'_r be neighbors of a_1, \dots, a_r in

R_1, \dots, R_r , respectively (if they exist). Then, for all j , the cube of R_j contains a hamiltonian path between a_j and a'_j ; call it q_j . First, we will construct in the cube of G a path between 3 and a_r ; the path is

$$3a'_1q_1a_1a'_2q_2a_2 \dots a'_r q_r a_r.$$

Let T_3 be the tree that contains 1, obtained from G after deleting the edge $a1$. Let b_1, \dots, b_s be the neighbors of 1 in T_3 . Let E_1, \dots, E_s be the trees obtained from T_3 by deleting, respectively, the edges $1b_1, \dots, 1b_s$ (E_j contains b_j). Let b'_1, \dots, b'_s be neighbors of b_1, \dots, b_s in E_1, \dots, E_s , respectively. For all $j = 1, \dots, s-1$, the cube of E_j contains a hamiltonian path, call it m_j , between b_j and b'_j . The cube of $E_s \cup \{1b_s\}$ also contains a hamiltonian path between b_s and 1; call it m_s .

In the cube of G , if we join the path

$$a_r b_1 m_1 b'_1 b_2 m_2 b'_2 \dots b_s m_s 1$$

to the path previously obtained between 3 and a_r , we get the desired path between 3 and 1.

Assuming that we have an orientation D of G that is not pushable to a hamiltonian one, we get that the 4-cycle $a123$ is good or bad, depending on the parity of $|T_1|$. Analogously, by picking another neighbor of a , say $1'$, we get that the cycle $a1'23$ is good or bad under D and because the path between a and 2 is the same in both cases, we get that the 4-cycles $a1'23$ and $a123$ are both either bad, or good. This implies, by Lemma 1, that the 4-cycle $a121'$ is good.

The next step of the proof is to show that in the cube of G there is a path $aw_1 \dots w_d 2$, such that the vertices of $G - \{a, 1, 2, 1', w_1, \dots, w_d\}$ form a path between 1 and $1'$ ($1' = a_1$). We take the path between a and 2 to be the same as the one we constructed at the beginning. In the cube of R_1 , there is a hamiltonian path between $1'$ and a'_1 ; call it g . The path between 1 and $1'$ is

$$1b'_1 m_1 b_1 \dots b'_s m_s b_s 3a'_2 q_2 a_2 \dots a'_r q_r a_r a'_1 g 1'.$$

Assuming that D is not pushable to a hamiltonian orientation, we get that the length of the path between a and 2 must be even. This implies that $|T_1|$ is odd. Since 3 is a randomly picked neighbor of a , we get that all the subtrees of G obtained after removing a and all its incident edges, have an odd number of vertices. Since $|G|$ is even, there must be an odd number of such subtrees.

The last step of the proof is to show that we can find in the cube of G a path $1z_1 \dots z_k 1'$ of odd length such that the vertices of $G - \{a, 2, 1, 1', z_1, \dots, z_k\}$ form a path between a and 2. This will imply, by Lemma 2, that D is pushable to a hamiltonian orientation, a contradiction.

Indeed, the path $1'b_1 m_1 b'_1 b_2 m_2 b'_2 \dots b_s m_s 1$ has odd length. To construct the path between a and 2, we introduce more notation. In R_1 , let $a'_1 = c_1, c_2, \dots, c_l$ be all the neighbors of $1'$. Let c'_1, c'_2, \dots, c'_l be neighbors, if they exist, of

c_1, \dots, c_l , respectively. Let S_1, \dots, S_l be the trees obtained from R_1 by deleting, respectively, the edges $1'c_1, \dots, 1'c_l$. For all j , the cube of S_j contains a hamiltonian path between c_j and c'_j ; call it o_j . The path between a and 2 is

$$ac'_1 o_1 c_1 \dots c'_l o_l c_l a_2 q_2 a'_2 \dots a_r q_r a'_r 3t'_1 P_1 t_1 \dots t'_{p+1} P_{p+1} t_{p+1},$$

where $t_{p+1} = 2$.

With this, the proof is complete.

4 Conclusions

We conclude the paper with a conjecture and a problem that could generalize the concept of a cyclable graph.

Conjecture 2 *The square of any two connected graph that is not an even cycle, is cyclable.*

Open Problem: Suppose we have a graph G with n vertices and p is an integer greater or equal to 2. We label the edges of G with elements from Z_p . We define a *push* at a vertex of G to be a replacement of the labels of all edges incident to that vertex, such that if an edge is labeled with k , after the push it will be labeled with $p - 1 - k$. G is said to be *p-cyclable* if for each labeling we can push a set of vertices such that the resulting labeling contains a hamiltonian cycle with the sum of the labels divisible by p . Is it true that for any p , there is an integer n_p , such that any graph with $n \geq n_p$ vertices and minimal degree $n/2 + f(p)$, where f is a suitable chosen function, is *p-cyclable*?

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