Improvements of Some Results on Cyclable Graphs

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- *First clue: hamiltonian problem.*

- *Classical hamiltonian results:*

<u>Degree results:</u>

  - any graph with minimal degree $n/2$ is hamiltonian. (Dirac '52)

  - $d_G(x)+d_G(y) \geq n$ for every nonadjacent vertices x and y in a graph G with n vertices, implies G hamiltonian. (Ore '61)

  - if $d_1 \leq \ldots \leq d_n$ is the degree sequence of a graph G with n vertices and $d_k \leq k-1 \leq n/2-1$ implies $d_{n+1-k} \geq n-k$, then G is hamiltonian. (Chvatal '72)

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<u>Degree results:</u>

  - any graph with minimal degree $n/2$ is hamiltonian. (Dirac '52)

  - $d_G(x) + d_G(y) \geq n + 1$ for every non-adjacent vertices x and y in a graph G with n vertices, implies G hamiltonian-connected.

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Results for powers of graphs:

cube of a connected graph is hamiltonian (Karaganis '68) and hamiltonian connected.

- square of a 2-connected graph is hamiltonian (Fleischner '72) and hamiltonian connected.

Results for powers of graphs:

- *Reminder:* given a graph G, we define G^k as follows: add in G an edge between any 2 vertices at distance at most k.

  cube of a connected graph is hamiltonian (Karaganis '68) and hamiltonian connected.

- square of a 2-connected graph is hamiltonian (Fleischner '72) and hamiltonian connected.

- *Reminder:* a graph is 2-connected if any 2 vertices can be joined by at least 2 edge-disjoint paths.

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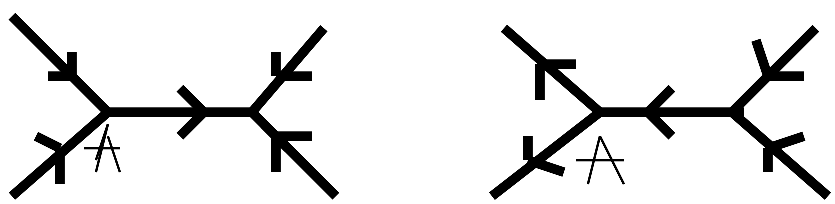
- stronger facts about the graph implied by the same conditions, such as:

  – panciclicity : graph contains cycles of all lengths.

  – cycle-extendability : any cycle with k vertices is contained in one with $k+1$ vertices.

Predef1 orientation of a graph = arbitrary assignment of direction to the edges of a simple graph

Predef2 push of a vertex of an oriented graph = reverse the orientations of all its incident edges



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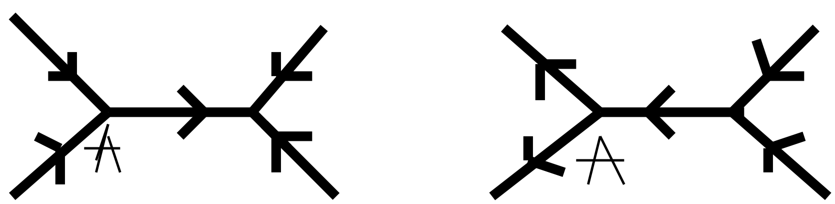


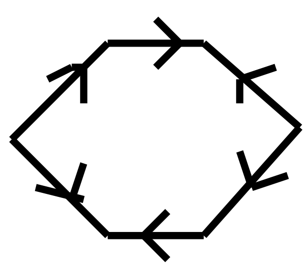
Def. A simple graph is said to be *cyclable* if for each orientation of its edges there is a subset of vertices that can be pushed, such that the resulting digraph has an oriented hamiltonian cycle.

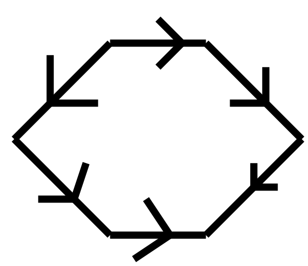
Remark: order in which vertices of a set are pushed does not matter.

Theorem For n odd, any hamiltonian graph on
K+S n vertices is cyclable.

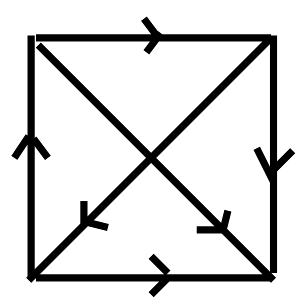
For n even, an orientation of a simple graph is pushable to one that contains a directed hamiltonian cycle if and only if it contains an oriented cycle passing through all the vertices with an even number of arrows clockwise.

Trivial: cyclable $\Rightarrow$ hamiltonian.





$K4$:the smallest hamiltonian graph that is not cyclable



Results of Klostermeyer and Soltes for powers of graphs:

- 4^{th} power of a connected graph with $\geq$5 vertices is cyclable.

- cube of any 2-connected graph with $\geq$5 vertices is cyclable.

- *Extremal results:*

  - cube of a path with $4k$ vertices is not cyclable.

  - square of an even cycle is not cyclable.

- *Conjecture:* square of a 3-connected graph with $\geq$5 vertices is cyclable.

Other results of Klostermeyer and Soltes:

G complete $k - partite$ graph with n vertices, $k \geq 2$; G cyclable $\Longleftrightarrow$ one of the following holds:

- the largest part of G contains $n/2$ vertices and n is divisible by 4.

- each part of$G$ contains less than $n/2$ vertices and $G \not\simeq K_{2,2,2}$.

In particular, $K_{n,n}$ cyclable $\Longleftrightarrow$ n even.

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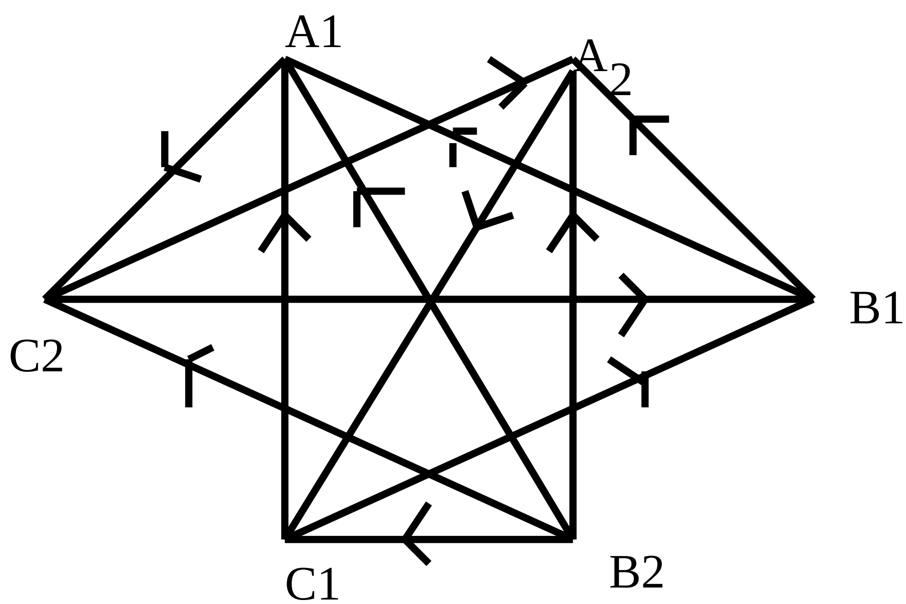
- *Conjecture:* square of any 2-connected graph that is not an even cycle, is cyclable.

Another (degree) result of K. and S.:

For a graph with $2n$ vertices, denote by $\gamma(n)$ the smallest positive integer, such that any graph with $2n$ vertices and minimal degree $\gamma(n)$ is cyclable. Then $n \leq \gamma(n) \leq n + 2$.
Otherwise formulated: any graph with minimal degree $n/2 + 2$ is cyclable.

My improvement:

Any graph with n vertices and minimal degree $n/2+1$ (n even), is cyclable, except for K_4 and $K_{2,2,2}$.



My conjectures:

- Any graph with n vertices such that the sum of the degrees of any two nonadjacent vertices is greater or equal to $n+2$ is cyclable (except perhaps for some particular cases).

- If $d_1 \leq \ldots \leq d_n$ is the degree sequence of a graph with n vertices and $d_{k-1} \leq k$ together with $2 \leq k \leq n/2$, implies $d_{n-k} \geq n-k+2$, then the graph is cyclable. (except perhaps for some particular cases).