Bessel and Volatility-Stabilized Processes

Irina Goia
Supervised by Professor Ioannis Karatzas

Doctoral Defense Presentation
Columbia University, January 24rd, 2009

References

- Banner, A., Fernholz, D., "Short-Term Arbitrage in Volatility-Stabilized Markets", Annals of Finance, to appear.
- Fernholz, E.R., "Stochastic Portfolio Theory", Springer-Verlag New York, 2002.
- Fernholz, E.R., "Short-Term Relative Arbitrage", unpublished note.
- Fernholz, E.R., Karatzas, I., "Relative Arbitrage in Volatility-Stabilized Markets", Annals of Finance **1**, 149-177, 2005.
- Fernholz, E.R., Karatzas, I., "Stochastic Portfolio Theory: An Overview", Handbook of Numerical Analysis, to appear.
- Fernholz, E.R., Karatzas, I., Kardaras, C., "Diversity and Relative Arbitrage in Equity Markets", Finance and Stochastics **9**, 1-27, 2005.

Outline of the talk

1. Some background from stochastic portfolio theory
2. The volatility-stabilized process (VSP): definition, representation in terms of Bessel processes, and known results
3. A study of the laws and moments of the market weights and coordinates of the VSP
4. Potential future work

Acknowledgements: I would like to thank my advisor, Prof. Ioannis Karatzas, and to Prof. Marc Yor for their substantial contribution towards the creation of this work. Special thanks to Julien Dubedat, Tomoyuki Ichiba and Jonathan Warren for their suggestions and technical help.

Some background from stochastic portfolio theory

Question: How to measure the cumulative volatility of a financial market driven by a multidimensional Brownian motion?
Answer: Excess growth-rate of the market portfolio.

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \sum_{k=1}^{d} \sigma_{ik}(t)dW_k(t)\right], \quad i = 1, \ldots, n.$$

$$\int_0^T \sum_{i=1}^{n} \left(|b_i(t)| + \sum_{k=1}^{d} \sigma_{ik}^2(t)\right) dt < \infty \;\; a.s.$$

$$d\left(\log X_i(t)\right) = \gamma_i(t)dt + \sum_{k=1}^{d} \sigma_{ik}(t)dW_k(t), \quad i = 1, \ldots, n,$$

$$\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t), \quad a_{ij}(t) = \sum_{k=1}^{n} \sigma_{ik}(t)\sigma_{jk}(t).$$

$\gamma_i(\cdot)$ is the growth rate of the i^{th} asset.

Portfolio process: $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))$, $\sum \pi_i = 1$, $\pi_i(t)$ is the proportion of wealth invested in the i^{th} asset at time t.

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)} = b^\pi(t)dt + \sum_{k=1}^{d} \sigma_k^\pi(t) dW_k(t), \text{ where}$$

$$b^\pi(t) := \sum_{i=1}^{n} \pi_i(t) b_i(t), \quad \sigma_k^\pi(t) := \sum_{i=1}^{n} \pi_i(t) \sigma_{ik}(t)$$

$$a^{\pi\pi}(t) := \sum_{k=1}^{d} (\sigma_k^\pi(t))^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$$

are the rate-of-return coefficients, the volatility coefficients and the variance of the portfolio.

$$d\left(\log Z^{\pi}(t)\right) = \gamma^{\pi}(t) + \sum_{k=1}^{d} \sigma_k^{\pi}(t) dW_k(t),$$

$$\gamma^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_*^{\pi}(t)$$

$$\gamma_*^{\pi}(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)a_{ij}(t)\pi_j(t)\right)$$

is called the excess growth rate of the portfolio $\pi(\cdot)$.

Market portfolio:

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}$$

$$\frac{dZ^\mu(t)}{Z^\mu(t)} = \frac{d\big(\sum X_i(t)\big)}{\sum X_i(t)}$$

$$\text{hence } \; Z^\mu(t) = \frac{z}{x}\big(\sum X_i(t)\big), \;\; \text{with } \; x := \sum X_i(0)$$

Relative arbritrage: We say that a portfolio rule $\pi(\cdot)$ is a relative arbitrage opportunity relative to a portfolio rule $\rho(\cdot)$ over the time horizon $[0, T]$ if $P\big[Z^\pi(T) \geq Z^\rho(T)\big] = 1$ and $P\big[Z^\pi(T) > Z^\rho(T)\big] > 0$ hold whenever the two portfolio rules start with the same initial fortune $Z^\pi(0) = Z^\rho(0) = z$.

Proposition: Suppose there exists a continuous, strictly increasing function $\Gamma : [0, \infty) \to [0, \infty)$ with $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$ and such that

$$\Gamma(t) \leq \int_0^t \gamma_*^\mu(s) ds < \infty, \;\; (\forall)\, 0 \leq t < \infty$$

holds almost surely. Then, with the entropy function $S(x) := -\sum_{j=1}^n x_j \log x_j$ and for any time horizon $[0, T]$ that satisfies: $\Gamma^{-1}\big(S\big(\mu(0)\big)\big) =: T_* < T < \infty$, there exists a sufficiently large real number $c > 0$ such that the portfolio rule $\pi_i(t) = \frac{c\mu_i(t) - \mu_i(t)\log\mu_i(t)}{c - \sum_{j=1}^n \mu_j(t)\log\mu_j(t)}$ $\;i = 1, \ldots, n$ is a strong arbitrage opportunity relative to the market portfolio; that is,

$$P\big[Z^\pi(T) > Z^\mu(T)\big] = 1.$$

Natural candidate: $\gamma_*^\mu(s) = ks^\alpha, \;\; \alpha \geq 0.$
One key feature of the VSP: $\gamma_*^\mu(s) =$constant.

The VSP: definition, representation of its coordinates in terms of Bessel processes, and known results

$$d\big(\log X_i(t)\big) = \frac{\alpha}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t), \quad i = 1, \ldots, n \text{ and } \alpha \geq 0$$

$$dX_i(t) = X_i(t)\left(\frac{1+\alpha}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t)\right)$$

A weak solution exists and is unique in distribution.

$$\gamma_*^\mu(t) = \gamma_* = \frac{n-1}{2}$$

$$\gamma^\mu(t) = \frac{(1+\alpha)n-1}{2} = \frac{mn}{4} - \frac{1}{2} =: \gamma > 0$$

$$\sum_{i=1}^n X_i(t) =: X(t) = x\exp(\gamma t + B_t), \quad B(t) := \sum_{i=1}^n \int_0^t \sqrt{\mu_i(s)}dW_i(s)$$

Starting trick: apply Itô's rule to $f(X_i(t))$, $f(x) = \sqrt{x}$.

$$\Lambda(t) := \int_0^t \left(\frac{X(s)}{4}\right) ds = \frac{x}{4} \int_0^t \exp(\gamma s + B_s) ds$$

$$\widehat{W}_i(t) := \int_0^{\Lambda^{-1}(t)} \sqrt{\Lambda'(u)} dW_i(u)$$

$$R_i(\cdot) := \sqrt{X_i\big(\Lambda^{-1}(\cdot)\big)}$$

$$R_i(t) = \sqrt{X_i(0)} + \int_0^t \frac{(m-1)ds}{2R_i(s)} + \widehat{W}_i(t)$$

$$\boxed{X_i(T) = R_i^2\big(\Lambda(T)\big), \quad X(T) = R^2\big(\Lambda(T)\big) \quad \Lambda^{-1}(t) = 4\int_0^t \frac{ds}{R^2(s)}}$$

Added bonus: new proof of Lamperti (Fernholz and Karatzas)

$$R^2\big(\Lambda(t)\big) = x\exp(\gamma t + B_t)$$

$$R^2\left(\frac{x}{4}\int_0^t \exp(\gamma s + B_s)ds\right) = x\exp(\gamma t + B_t)$$

$$R\left(x\int_0^\theta \exp\big(2(2\gamma s + \tilde{B}_s)\big)ds\right) = \sqrt{x}\exp\left(2\gamma\theta + \tilde{B}_\theta\right),$$

$$\text{where} \quad \tilde{B}(\cdot) := \frac{1}{2}B(4\cdot)$$

Long term behaviour of the VSP

Proposition: For a market model based on the VSP the growth rate for the entire market and for the biggest stock are computed as

$$\gamma = \lim_{t\to\infty}\left(\frac{1}{t}\log X(t)\right) = \lim_{t\to\infty}\left(\frac{1}{t}\log X_{(1)}(t)\right) = \lim_{t\to\infty}\left(\frac{1}{t}\log X_i(t)\right) \; a.s$$

The first two limits hold for any $\alpha \geq 0$, the last one only for $\alpha > 0$.

Key ingredient of the proof: LLN for the Bessel process (Yor, Zani)

Theorem: Let $\left(R_t^{(\nu)}, t \geq 0\right)$ be a Bessel process starting from $R_0^{(\nu)} \neq 0$ a.s., with dimension $d > 2$ and index $\nu := \frac{d}{2} - 1 > 0$. We have the following:

$$\frac{1}{\log t}\int_0^t \frac{ds}{(R_s^{(\nu)})^2} \underset{t\to\infty}{\longrightarrow} \frac{1}{d-2} = \frac{1}{2\nu} \text{ a.s. and in } L^p$$

Proposition: For every $u \in [0, \infty)$, $i = 1, \ldots, n$ and $\delta \in (0,1)$ we have

$$\lim_{u \to \infty} P\left[\mu_i\left(\Lambda^{-1}(u)\right) \leq 1 - \delta\right] = 1 - \delta^{n-1}$$

Added bonus: a market model based on the VSP is not diverse: there is no number $\delta \in (0,1)$ such that

$$P\left[\max_{1 \leq i \leq n} \mu_i(t) < 1 - \delta,\ (\forall)\ 0 \leq t < \infty\right] = 1.$$

Not known whether there is no δ such that:
$\frac{1}{T}\int_0^T \mu_{(1)}(t)dt < 1 - \delta$ (weak diversity condition)
Finally: a market model based on the VSP does not satisfy the boundedness away from infinity of the volatility matrix $\xi^t \sigma(t)\sigma^t(t)\xi \leq M||\xi||^2$ $(\forall)$ $t \geq 0$ and $\xi \in \mathbb{R}^n$. (Proof involves the Laplace transform of the squared Bessel process.)

A study of the laws and moments of the market weights and stock prices

The skew-product of CIR processes and the multidimensional Jacobi process

Consider a family of CIR processes, Q_i, with parameter sets (δ_i, b, η) $(dQ_i(t) = (\delta_i - bQ_i(t))dt + \eta\sqrt{X_t^+}dW_t)$. Then $Q := \sum_{i=1}^n Q_i$ is a CIR process with parameter set (δ, b, η), where $\delta := \sum_{i=1}^n \delta_i$. Define the clock $C(t) := \int_0^t \frac{ds}{Q(s)}$. Then $\frac{Q_i}{Q} = Y_i(C(t)),\ \ i = 1, \ldots, n$ and the vector process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ is a multidimensional diffusion that satisfies the following system of SDEs:

$$dY_i(t) = (\delta_i - \delta Y_i(t))\,dt + \eta(1 - Y_i(t)\sqrt{Y_i(t)}dW_i(t) -$$

$$-\eta Y_i(t)\sum_{j\neq i}\sqrt{Y_j(t)}dW_j(t), \quad Y_i(0) = \frac{R_i^2(0)}{R^2(0)}.$$

Each of the processes $Y_i(\cdot)$ is a one-dimensional Jacobi diffusion:

$$dY_i(t) = (\delta_i - \delta Y_i(t))\,dt + \eta\sqrt{Y_i(t)(1 - Y_i(t))}dB_i(t).$$

The market weights as a multidimensional Jacobi process

$$\mu_i(T) = \frac{X_i(T)}{X(T)} = \frac{R_i^2\left(\Lambda(T)\right)}{R^2\left(\Lambda(T)\right)} \quad i = 1, \ldots, n$$

$$R_i^2\left(\Lambda(T)\right) = R^2\left(\Lambda(T)\right) Y_{i\left(\int_0^{\Lambda(T)} \frac{ds}{R_s^2}\right)}$$

$$\mu_i(T) = Y_i\left(\frac{T}{4}\right) \quad i = 1, \ldots, n$$

Facts about Jacobi and CIR processes

- CIR and Bessel are examples of Galton-Watson processes with immigration (Kawazu, Watanabe):
  $E\left[e^{-\lambda X(t)}\right] = \phi(t,\lambda)e^{-x\psi(t,\lambda)}, \;\; X(0) = x$
- transition density of Jacobi is not known
- spectral theory of the infinitesimal generator of Jacobi has been studied (Wong,1964); eigenvalues are the Jacobi polynomials
- moments of Jacobi can be recursively computed
- invariant distribution of multidimensional Jacobi is the Dirichlet distribution
  (p.d.f $\mu(\overline{x}) = x_1^{\frac{2\delta_1}{\eta^2}-1} \cdot \cdots \cdot x_n^{\frac{2\delta_n}{\eta^2}-1}, \;\; \overline{x} \in \Delta^n$)
- Jacobi has been studied by Forman, Sorensen, Larsen; Gouriéroux, Valéry, Jasiak

The law of market weights via the spectral representation of the transition density of a diffusion (Karlin, Taylor)

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \ \ X(0) = x_0$$

diffusion on $I = [l, r]$, l and r exit or reflecting boundaries.

$$P[X(t) \in dy]/dy = p_t(x, y) = m(y) \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y) \pi_k$$

$$s(x) = e^{-\int_{x_0}^{x} dy \frac{2b(y)}{\sigma^2(y)}} \text{ and } m(x) = \frac{1}{\sigma^2(x)s(x)}$$

ϕ_k and λ_k are the eigenvalues and corresponding eigenfunctions of the infinitesimal generator $(\mathcal{L}f)(x) = \frac{1}{2}\sigma^2(x)f^{''}(x) + b(x)f^{'}(x)$

$$\pi_k := \frac{1}{\int_l^r dy m(y) \phi_k^2(y)}$$

What are π_k, λ_k, and ϕ_k for the Jacobi diffusion?

$$\lambda_k = k(nm + 2k - 2), \quad k \geq 0 \quad \pi_k = \pi_0 = \frac{\Gamma\left(\frac{mn}{2}\right)}{2^{\frac{mn}{2}-2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)}$$

$$P_k(y) = \left[\frac{\Gamma\left(k+\frac{m}{2}\right)\left(2k+\frac{nm}{2}-1\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)}{k!\Gamma\left(\frac{nm}{2}+k-1\right)\Gamma\left(\frac{nm}{2}\right)\Gamma\left(\frac{m(n-1)}{2}+k\right)}\right]^{\frac{1}{2}} \cdot$$

$$\cdot \sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\Gamma\left(\frac{mn}{2}+k+i-1\right)}{\Gamma\left(\frac{m}{2}+i\right)}y^i$$

$$\mathbf{P}[\mu_{\mathbf{i}}(\mathbf{T}) \in \mathbf{dy}]/\mathbf{dy} =$$

$$\frac{y^{\frac{m}{2}-1}(1-y)^{\frac{(n-1)m}{2}-1}\Gamma\left(\frac{mn}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m(n-1)}{2}\right)}\sum_{k=0}^{\infty}e^{-k(nm+2k-2)\frac{T}{4}}P_k(\mu_i(0))P_k(y)$$

The moments of market weights

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad \mathcal{L}h = -\lambda h$$

$$E\left[h(X(t))\right] = e^{-\lambda t} h(x), \text{ under certain technical assumptions}$$

If $dY(t) = (m - mnY(t))dt + 2\sqrt{Y(t)(1 - Y(t))}dB(t), \quad Y(0) = y$

$$E\left[Q_k(Y(t))\right] = Q_k(Y(0))e^{-k(mn+2k-2)t}$$

$$x_j := E\left[Y^j(t)\right], \quad Q_k(x) = \sum_{j=0}^{k} a_{kj} x^j, \quad \text{where}$$

$$a_{kj} = (-1)^j \binom{k}{j} \frac{\Gamma\left(\frac{mn}{2} + k + j - 1\right)}{\Gamma\left(\frac{m}{2} + j\right)}$$

$$\sum_{j=1}^{k} a_{kj}\left(x_j - y^j e^{-k(mn+2k-2)t}\right) = a_{k0}\left(e^{-k(mn+2k-2)t-1}\right)$$

The mean and variance of the market weights

$$E\left[\mu_i(T)\right] = \frac{R_i^2(0)}{R^2(0)} e^{-mn\frac{T}{4}} + \frac{1}{n}\left(1 - e^{-mn\frac{T}{4}}\right)$$

$$\mathrm{Var}\left[\mu_i(T)\right] = \mathcal{P}_T\left(\frac{R_i^2(0)}{R^2(0)}\right)$$

$$\mathcal{P}_T(x) = A(T)x^2 + B(T)x + G(T) - F(T), \quad A(T) = e^{-2mn\frac{T}{4}}(e^{-T} - 1)$$

Several observations

- $\lim_{T\to\infty} E\left[\mu_i(T)\right] = \frac{1}{n}$ and $\lim_{T\to\infty} E\left[\mu_i^2(T)\right] = \frac{m+2}{n(mn+2)}$
- the p^{th} moment of the i^{th} market weight is a polynomial of degree p in $\frac{R_i^2(0)}{R^2(0)}$
- the monotonicity of $E\left[\mu_i(T)\right]$ is easy to establish
- for any $T > 0$ the ordering of the variances of the market weights at time T is the same as the ordering of the set of initial data $\{R_1(0), \ldots, R_n(0)\}$

Theorem: The law of a coordinate of the VSP

$$\frac{P\left[\sqrt{X_i(T)} \in dr_i\right]}{dr_i} =$$

$$2r_i \int_0^\infty \int_{r_i}^\infty p^{(i)}_{\frac{T}{4}}\left(\frac{R_i^2(0)}{R^2(0)}, \frac{r_i^2}{r^2}\right) \frac{r^{\gamma-1}}{tR(0)^\gamma} e^{-\frac{\gamma^2 T}{8}} e^{-\frac{r^2+R^2(0)}{2t}} \theta_{\frac{rR(0)}{t}}\left(\frac{T}{4}\right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)}$$

$$\text{where } \; I_{t,T,R(0)}(\gamma) := \int_0^\infty z^\gamma \theta_{\frac{zR(0)}{t}}\left(\frac{T}{4}\right) e^{-\frac{z^2}{2t}} dz$$

$$\theta_r(u) := \frac{re^{\frac{\pi^2}{2u}}}{\sqrt{2\pi^3 u}} \int_0^\infty e^{-\frac{y^2}{2u}} e^{-r\cosh y} (\sinh y) \sin\left(\frac{\pi y}{u}\right) dy$$

Theorem: The joint law of the coordinates of the VSP

$$P\left[\sqrt{X_1(T)} \in dr_1, \ldots, \sqrt{X_n(T)} \in dr_n\right] =$$

$$\frac{e^{-\frac{\gamma^2 T}{8}}}{R(0)^\gamma} 2^{n-1} r_1 \ldots r_n (r_1^2 + \cdots + r_n^2)^{\frac{\gamma+2-2n}{2}} p_{\frac{T}{4}} \left(\frac{R_1^2(0)}{R^2(0)}, \ldots, \frac{R_n^2(0)}{R^2(0)}; \frac{r_1^2}{\sum r_i^2}, \ldots, \frac{r_n^2}{\sum r_i^2} \right) \cdot$$

$$\cdot \int_0^\infty e^{-\frac{r_1^2+\cdots+r_n^2+R^2(0)}{2t}} \theta_{\frac{\sqrt{\sum r_i^2} R(0)}{t}} \left(\frac{T}{4}\right) \frac{I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t},$$

where $\gamma = \frac{mn}{2} - 1$

The main ideas of the proofs

- $E^{\gamma}\left[e^{-\lambda\int_0^t \frac{ds}{R^2(s)}}\middle| R(t)=r\right]=\frac{I_{\sqrt{2\lambda+\gamma^2}}}{I_{\gamma}}\left(\frac{rR(0)}{t}\right)$

$$R(t)=R(0)+\left(B(t)-\gamma\int_0^t\frac{ds}{R(s)}\right)+\left(\gamma+\frac{1}{2}\right)\int_0^t\frac{ds}{R(s)}$$

  Idea encountered at: Warren & Yor, Linetsky, Göing-Jaeschke

- the above Laplace transform can be inverted using
  $I_{|\gamma|}(r)=\int_0^{\infty}e^{-\frac{\gamma^2 u}{2}}\theta_r(u)du,\ I_v(z)=\left(\frac{z}{2}\right)^v\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\Gamma(v+k+1)}$

- start by writing

$$P\left[\sqrt{X_i(T)}\in dr_i\right]=P\left[R_i(\Lambda(T))\in dr_i\right]=\int_0^{\infty}P\left[R_i(t)\in dr_i|\ \Lambda(T)=t\right]P\left[\Lambda(T)\in dt\right]$$

- express $E\left[f\left(R_i^2(t)\right)g\left(C(t)\right)\right]$ in two different ways...

A formula for the moments of the coordinates of the VSP

$$M_p := E\left[X_i^p(T)\right]$$

$$M_p = \left[\int_0^1 j^p p_{\frac{T}{4}}^{(i)}\left(\frac{R_i^2(0)}{R^2(0)}, j\right) dj\right] \cdot$$

$$\cdot \frac{e^{-\frac{\gamma^2 T}{8}}}{R^{\gamma}(0)} \int_0^{\infty} e^{-\frac{R^2(0)}{2t}} \frac{I_{t,T,R(0)}(2p+\gamma+1) I_{t,T,R(0)}(\gamma-1)}{I_{t,T,R(0)}(\gamma+1)} \frac{dt}{t}$$

Potential future work

- compare the distribution of a coordinate of the VSP with the log-normal distribution, using total variation distance
- spectral theory of the multidimensional Jacobi process: are eigenvalues the multivariate Jacobi polynomials? can we compute mixed moments? Such a theory would be useful towards making progress on the 'shortest time to beat the market' open question
- monotonicity with respect to T of $M_p := E\left[X_i^p(T)\right]$, at least for $p = 1$
- study the more general volatility-driven model given by the system on SDEs below

$$d\left(\log X_i(t)\right) = \frac{\alpha}{2\left(\mu_i(t)\right)^{2\beta}} + \frac{\sigma}{\left(\mu_i(t)\right)^{\beta}} dW_i(t), \quad i = 1, \ldots, n.$$